Dr. Z.'s Calc5 Lecture 3 Handout: Translation Theorems for the Laplace Transform

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Part A: Translation along the s-axis $s \to s-a$.

Important Formulas

If, as usual $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

and, of course

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$$

Problem 3.1: Find $\mathcal{L}\{t^2 e^{-3t}\}$.

Sol. Here $f(t) = t^2$ and we know from tables that $F(s) = \frac{2}{s^3}$. In this problem $a = -3$ so $\mathcal{L}\{t^2 e^{-3t}\} = \frac{2}{(s-(-3))^3} = \frac{2}{(s+3)^3}$.

Ans. to 3.1: $\frac{2}{(s+3)^3}$

Problem 3.2: Find $\mathcal{L}\{e^{2t}\cos 5t\}$.

Sol. Here $f(t) = \cos 5t$ and $a = 2$. We know from tables that $F(s) = \frac{s}{s^2+25}$. In this problem $a = 2$ so $\mathcal{L}\{e^{2t}\cos 5t\} = \frac{s-2}{(s-2)^2+25}$.

Ans. to 3.2: $\frac{s-2}{(s-2)^2+25}$.

Problem 3.3: Find

$$\mathcal{L}^{-1}\{\frac{1}{s^2-4s+13}\} \quad .$$

Sol.: Since $s^2-4s+13$ has **complex roots**, we must *complete the square.*

$$s^2-4s+13 = (s-2)^2+9 \quad .$$

So

$$\mathcal{L}^{-1}\{\frac{1}{s^2-4s+13}\} = \mathcal{L}^{-1}\{\frac{1}{(s-2)^2+9}\} \quad .$$

Now we do *pattern-recognition.* From the tables:

$$\mathcal{L}^{-1}\{\frac{1}{s^2+9}\} = \frac{1}{3}\sin 3t \quad ,$$

So

$$\mathcal{L}^{-1}\{\frac{1}{(s-2)^2+9}\} = \frac{1}{3}e^{2t}\sin 3t \quad .$$

Ans. to 3.3: $\frac{1}{3}e^{2t}\sin 3t$.

It is much easier, from now on, to remember (or know how to use) the following, that follow from the tables of Lecture 1 and Lecture 2.

Most Important Laplace Transforms

In real life, people (usually) don't compute the Laplace Transform from stratch, they use tables. The most important items are:

$$(a) \quad \mathcal{L}\{t^k e^{at}\} = \frac{k!}{(s-a)^{k+1}} \quad (k = 1, 2, 3, ...),$$

$$(b) \quad \mathcal{L}\{e^{at} \sin kt\} = \frac{k}{(s-a)^2 + k^2} \quad ,$$

$$(c) \quad \mathcal{L}\{e^{at} \cos kt\} = \frac{s-a}{(s-a)^2 + k^2} \quad ,$$

Most Important Inverse Laplace Transforms

$$(a') \quad \mathcal{L}^{-1}\{\frac{1}{(s-a)^k}\} = \frac{t^{k-1}e^{at}}{(k-1)!} \quad (k = 1, 2, 3, ...),$$

$$(b') \quad \mathcal{L}^{-1}\{\frac{1}{(s-a)^2 + k^2}\} = \frac{e^{at} \sin kt}{k} \quad ,$$

$$(c') \quad \mathcal{L}^{-1}\{\frac{s-a}{(s-a)^2 + k^2}\} = e^{at} \cos kt \quad ,$$

Problem 3.4: Use the Laplace Transform to solve the following initial-value problem.

$$y' - 2y = e^{2t} \quad , \quad y(0) = 3.$$

Solution: Putting $\mathcal{L}\{y\} = Y$, we have $\mathcal{L}\{y'\} = sY - y(0) = sY - 3$.

Applying $\mathcal{L}$ to the equation gives:

$$\mathcal{L}\{y' - 2y\} = \mathcal{L}\{e^{2t}\} \quad ,$$

$$sY - 3 - 2Y = \frac{1}{s-2} \quad .$$

Solving for Y gives

$$(s-2)Y - 3 = \frac{1}{s-2} \quad ,$$

$$(s-2)Y = 3 + \frac{1}{s-2} = \frac{3s-5}{s-2} \quad ,$$

$$Y = \frac{3s-5}{(s-2)^2} \quad .$$

You can do **partial fraction** but it is faster to write $s = (s-2)+2$, and get

$$Y = \frac{3((s-2)+2)-5}{(s-2)^2} = \frac{3(s-2)+1}{(s-2)^2} = \frac{3}{s-2} + \frac{1}{(s-2)^2} \quad .$$

So

$$y = \mathcal{L}^{-1}\{\frac{3}{s-2}\} + \mathcal{L}^{-1}\{\frac{1}{(s-2)^2}\} = 3e^{2t} + te^{2t} = (3+t)e^{2t} \quad .$$

Ans. to 3.4: $y(t) = (3+t)e^{2t}$.

Problem 3.5: Use the Laplace Transform to solve the following initial-value problem.

$$y'' - 4y' + 4y = t^3e^{2t} \quad , \quad y(0) = 0 \quad , \quad y'(0) = 0 \quad .$$

Solution of 3.5: Applying $\mathcal{L}$:

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{t^3e^{2t}\} \quad , \quad y(0) = 0 \quad , \quad y'(0) = 0 \quad .$$

Now, the **translation**:

$$\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0) = s^2Y - 0 - 0 = s^2Y \quad , \quad \mathcal{L}\{y'\} = sY - y(0) = sY - 0 = sY \quad , \quad \mathcal{L}\{y\} = Y \quad ,$$

and (using (a) above with $k = 3, a = 2$):

$$\mathcal{L}\{t^3e^{2t}\} = \frac{3!}{(s-2)^{3+1}} = \frac{6}{(s-2)^4} \quad .$$

So we get the following **algebraic equation** for $Y = Y(s)$:

$$s^2Y - 4sY + 4Y = \frac{6}{(s-2)^4} \quad .$$

Solving it, we have

$$(s^2 - 4y + 4)Y = \frac{6}{(s-2)^4}$$

$$(s-2)^2Y = \frac{6}{(s-2)^4}$$

$$Y = \frac{6}{(s-2)^6} = 6\frac{1}{(s-2)^6} \quad .$$

So

$$y(t) = 6\mathcal{L}^{-1}\{\frac{1}{(s-2)^6}\} \quad .$$

Now using (a') with $a = 2, k = 6$ we get:

$$\mathcal{L}^{-1}\{\frac{1}{(s-2)^6}\} = \frac{t^{6-1}e^{2t}}{(6-1)!} = \frac{t^5e^{2t}}{120} \quad ,$$

yielding:

$$y(t) = 6\frac{t^5 e^{2t}}{120} = \frac{t^5 e^{2t}}{20} \quad .$$

Ans. to 3.5: $y(t) = \frac{t^5 e^{2t}}{20}$

Part B: Translation along the t-axis $f(t) \to f(t-a)\mathcal{U}(t-a)$.

The functions $f(t)$ that the Laplace Transform transforms, start at time $t = 0$, what happened in the past ($t < 0$) is irrelevant, since

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \quad ,$$

and the integral stats at $t = 0$ and goes for ever.

But what if the function starts later, at $t = a$?, for some a in the future? For that we have the **atomic** function that can be used to express functions that start later. It is called the **unit step function** or the **Heaviside function**.

VERY IMPORTANT DEFINITION

$$\mathcal{U}(t) = \begin{cases} 0, & \text{if } t < 0; \\ 1, & \text{if } t \geq 0. \end{cases}$$

It is the function that is 0 until $t = 0$ and **all of a sudden** starts being 1 for ever after! Of course it is **discontinuous** at $t = 0$.

But what if you are born at a later time, $t = a$, then we have the **translation** of $\mathcal{U}(t)$:

$$\mathcal{U}(t-a) = \begin{cases} 0, & \text{if } t < a; \\ 1, & \text{if } t \geq a. \end{cases}$$

What's nice about these translates of $\mathcal{U}$ is that we can rewrite annoying functions given by different formulas for different intervals using $\mathcal{U}(t-a)$ for various a. For example, if a function $f(t)$ is born at $t = a$ and dies at $t = b$, so it is

$$f(t) = \begin{cases} 0, & \text{if } t < a; \\ g(t), & \text{if } a \leq t \leq b \ , \\ 0, & \text{if } t \geq b. \end{cases} \quad ,$$

then we get rid of the annoying { notation, and write

$$f(t) = g(t) \quad [\quad \mathcal{U}(t-a) - \mathcal{U}(t-b) \quad] \quad .$$

If we have several pieces, then we repeat this for every interval, and add them up. Of course, from the perspective of the Laplace Transform $\mathcal{U}(t) = 1$, and $\mathcal{U}(t-\infty) = 0$.

Problem 3.6: Express in terms of $\mathcal{U}(t-a)$ the following function

$$f(t) = \begin{cases} t^2, & \text{if } 0 \le t < 2; \\ 0, & \text{if } t \ge 2. \end{cases} \quad ,$$

Solution: $f(t) = t^2(\mathcal{U}(t-0) - \mathcal{U}(t-2)) + 0[\mathcal{U}(t-\infty) - \mathcal{U}(t-2)] = t^2(1 - \mathcal{U}(t-2)) = t^2 - t^2\mathcal{U}(t-2)$.

Ans. to Problem 3.6: $f(t) = t^2 - t^2\mathcal{U}(t-2)$.

Problem 3.7: Express in terms of $\mathcal{U}(t-a)$ the following function

$$f(t) = \begin{cases} t^2, & \text{if } 0 \le t < 3; \\ t^3, & \text{if } 3 \le t \le 7; \\ -t^5, & \text{if } 7 \le t < \infty. \end{cases} \quad ,$$

Solution:

$$f(t) = t^2(\mathcal{U}(t-0) - \mathcal{U}(t-3)) \; + \; t^3(\mathcal{U}(t-3) - \mathcal{U}(t-7)) \; - t^5(\mathcal{U}(t-7) - \mathcal{U}(t-\infty)) =$$

$t^2(1 - \mathcal{U}(t-3)) + t^3(\mathcal{U}(t-3) - \mathcal{U}(t-7)) - t^5(\mathcal{U}(t-7) - 0) = t^2 + (t^3 - t^2)\mathcal{U}(t-3) - (t^3 + t^5)\mathcal{U}(t-7)$

Ans. to 3.7: $f(t) = t^2 + (t^3 - t^2)\mathcal{U}(t-3) - (t^3 + t^5)\mathcal{U}(t-7)$.

If $f(t)$, that starts at $t = 0$ is *translated* to $t = a$ (i.e. we postpone the movie to $t = a$) we get $f(t-a)\mathcal{U}(t-a)$.

Important Formulas

If $F(s) = \mathcal{L}\{f(t)\}$ and $a > 0$ then

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s) \quad .$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a) \quad .$$

Problem 3.8 Compute

$$\mathcal{L}\{t^2\mathcal{U}(t-1)\} \quad .$$

Solution: First we have to express t^2 in terms of $(t-1)$:

$$t^2 = ((t-1)+1)^2 = (t-1)^2 + 2(t-1) + 1 \quad .$$

So

$$\mathcal{L}\{t^2\mathcal{U}(t-1)\} = \mathcal{L}\{((t-1)^2 + 2(t-1) + 1)\mathcal{U}(t-1)\} = \mathcal{L}\{(t-1)^2\mathcal{U}(t-1)\} + 2\mathcal{L}\{(t-1)\mathcal{U}(t-1)\} + \mathcal{L}\{\mathcal{U}(t-1)\}$$

$$= e^{-s}\frac{2!}{s^3} + 2e^{-s}\frac{1!}{s^2} + e^{-s}\frac{1}{s} = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) \quad .$$

Ans. to 3.8: $\mathcal{L}\{t^2\mathcal{U}(t-1)\} = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$.

Problem 3.9 Compute

$$\mathcal{L}^{-1}\{\frac{e^{-3s}}{s^3}\} \quad .$$

Solution: Here $F(s) = \frac{1}{s^3}$ and $a = 3$ in the general formula

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a) \quad .$$

We have $f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{t^2}{2}$, so

$$\mathcal{L}^{-1}\{\frac{e^{-3s}}{s^3}\} = f(t-3)\mathcal{U}(t-3) = \frac{(t-3)^2}{2}\mathcal{U}(t-3) \quad .$$

Ans. to 3.9: $\frac{(t-3)^2}{2}\mathcal{U}(t-3)$

Often, in applications to solving differential functions, the **driving force**, i.e. the *right side*, is a function given by different formulas for different intervals, i.e. one of those annoying { functions. Now we can handle them, thanks for the above formulas involving $\mathcal{U}(t-a)$ and e^{-as}.

Problem 3.10: Use the Laplace Transform to solve the following initial-value problem.

$$y' + y = f(t) \quad , \quad y(0) = 0 \quad , \quad where f(t) = \begin{cases} 1, & \text{if } 0 \le t < 1; \\ -1, & \text{if } 1 \le t < \infty. \end{cases} \quad .$$

Solution: We first express $f(t)$ in terms of $\mathcal{U}(t-a)$'s.

$f(t) = 1(\mathcal{U}(t-0) - \mathcal{U}(t-1)) + (-1)(\mathcal{U}(t-1) - \mathcal{U}(t-\infty)) = 1 - \mathcal{U}(t-1) - \mathcal{U}(t-1) + 0 = 1 - 2\mathcal{U}(t-1) \quad .$

So we have to solve

$$y' + y = 1 - 2\mathcal{U}(t-1), y(0) = 0 \quad .$$

Applying $\mathcal{L}$:

$$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{1\} - 2\mathcal{L}\{1 \cdot \mathcal{U}(t-1)\}$$

$\mathcal{L}\{y'\} = sY - y(0) = sY - 0 = sY$, $\mathcal{L}\{y\} = Y$, $\mathcal{L}\{1\} = \frac{1}{s}$, $\mathcal{L}\{1 \cdot \mathcal{U}(t-1)\} = \frac{e^{-s}}{s}$. So:

$$sY + Y = \frac{1 - 2e^{-s}}{s} \quad ,$$

$$(s+1)Y = \frac{1 - 2e^{-s}}{s} \quad ,$$

$$Y = \frac{1 - 2e^{-s}}{s(s+1)} \quad .$$

Partial fractions: $\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{A(s+1)+Bs}{s(s+1)}$

$$1 = A(s+1) + Bs \quad .$$

Covenient values: $s = 0 : 1 = A(0+1) + B(0)$ so $A = 1$, $s = -1 : 1 = A(-1+1) + B(-1)$ so $B = -1$. So $\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$ Writing Y as $(1 - 2e^{-s})F(s)$ where $F(s) = \frac{1}{s(s+1)}$ we have

$$f(t) = \mathcal{L}^{-1}\{\frac{1}{s} - \frac{1}{s+1}\} = 1 - e^{-t} \quad .$$

Finally

$$y(t) = \mathcal{L}^{-1}\{\frac{1}{s(s+1)}\} - 2\mathcal{L}^{-1}\{\frac{e^{-s}}{s(s+1)}\} = f(t) - 2f(t-1)\mathcal{U}(t-1) = (1-e^{-t}) - 2(1-e^{-(t-1)})\mathcal{U}(t-1) \quad .$$

Ans. to 3.10: $y(t) = (1 - e^{-t}) - 2(1 - e^{-(t-1)})\mathcal{U}(t-1)$.