Dr. Z.'s Calc5 Lecture 20 Handout: Fourier Integrals

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Important Definition: Fourier Integral Representation

The **Fourier Integral Representation** of a function $f(x)$ defined on the real line $(-\infty, \infty)$ is given by

$$\frac{1}{\pi}\int_0^\infty [\, A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\,]\, d\alpha \quad ,$$

where

$$A(\alpha) = \int_{-\infty}^{\infty} f(x)\cos\alpha x\, dx$$

$$B(\alpha) = \int_{-\infty}^{\infty} f(x)\sin\alpha x\, dx$$

Important Theorem: If $f(x)$ is well-behaved and $\int_{-\infty}^{\infty} |f(x)|\, dx$ is finite, then the Fourier integral of $f(x)$ "converges" to it (in the sense of an improper integral over $(0,\infty)$), if $f(x)$ is continuous. If it is piece-wise continuous, it converges to it everywhere except at the discontinuities (in which case it is the average of the limits from the left and right).

Problem 20.1: Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 0, & \text{if } x < -1; \\ 1, & \text{if } -1 \le x < 2; \\ 0, & \text{if } x > 2; \end{cases}$$

Solution:

$$A(\alpha) = \int_{-\infty}^{\infty} f(x)\cos\alpha x\, dx = \int_{-\infty}^{-1} f(x)\cos\alpha x\, dx + \int_{-1}^{2} f(x)\cos\alpha x\, dx + \int_{2}^{\infty} f(x)\cos\alpha x\, dx$$

$$= \int_{-\infty}^{-1} 0\cdot\cos\alpha x\, dx + \int_{-1}^{2} 1\cdot\cos\alpha x\, dx + \int_{2}^{\infty} 0\cdot\cos\alpha x\, dx = 0 + \int_{-1}^{2}\cos\alpha x\, dx + 0 = \int_{-1}^{2}\cos\alpha x\, dx$$

$$= \frac{\sin\alpha x}{\alpha}\Big|_{-1}^{2} = \frac{\sin(\alpha(2)) - \sin(\alpha(-1))}{\alpha} = \frac{\sin(2\alpha) - \sin(-\alpha)}{\alpha} = \frac{\sin 2\alpha + \sin\alpha}{\alpha} \quad .$$

Analogously,

$$B(\alpha) = \int_{-\infty}^{\infty} f(x)\sin\alpha x\, dx = \int_{-\infty}^{-1} f(x)\sin\alpha x\, dx + \int_{-1}^{2} f(x)\sin\alpha x\, dx + \int_{2}^{\infty} f(x)\sin\alpha x\, dx$$

$$= \int_{-\infty}^{-1} 0\cdot\sin\alpha x\, dx + \int_{-1}^{2} 1\cdot\sin\alpha x\, dx + \int_{2}^{\infty} 0\cdot\sin\alpha x\, dx = 0 + \int_{-1}^{2}\sin\alpha x\, dx + 0 = \int_{-1}^{2}\sin\alpha x\, dx$$

$$= -\frac{\cos\alpha x}{\alpha}\Big|_{-1}^{2} = -\frac{\cos(\alpha(2)) - \cos(\alpha(-1))}{\alpha} = -\frac{\cos(2\alpha) - \cos(-\alpha)}{\alpha} = -\frac{\cos 2\alpha - \cos\alpha}{\alpha} \quad .$$

Putting both $A(\alpha)$ and $B(\alpha)$ into the formula for the Fourier Integral of $f(x)$,

$$\frac{1}{\pi}\int_0^\infty [\, A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\,]\, d\alpha \quad ,$$

we get:

$$\frac{1}{\pi}\int_0^\infty [\, \frac{\sin 2\alpha + \sin\alpha}{\alpha}\cos\alpha x - \frac{\cos 2\alpha - \cos\alpha}{\alpha}\sin\alpha x\,]\, d\alpha \quad .$$

This is a **correct** answer, but using **trig. identities**, we can get a nicer answer:

$$\frac{1}{\pi}\int_0^\infty \frac{\sin 2\alpha\cos\alpha x - \cos 2\alpha\sin\alpha x) + (\sin\alpha\cos\alpha x + \cos\alpha\sin\alpha x)}{\alpha}\, d\alpha$$

$$= \frac{1}{\pi}\int_0^\infty \frac{(\sin(2\alpha - \alpha x) + \sin(\alpha + \alpha x)}{\alpha}\, d\alpha = \frac{1}{\pi}\int_0^\infty \frac{\sin(\alpha(2-x)) + \sin(\alpha(1+x))}{\alpha}\, d\alpha$$

Ans. to 20.1: The Fourier integral representation of $f(x)$ of the problem is $\frac{1}{\pi}\int_0^\infty \frac{\sin(\alpha(2-x))+\sin(\alpha(1+x))}{\alpha}\, d\alpha$.

Important Definition: Fourier Cosine Integral

The **Fourier Cosine Integral** of an **even** function $f(x)$ defined on the real line $(-\infty, \infty)$ is the **cosine integral**

$$\frac{2}{\pi}\int_0^\infty A(\alpha)\cos\alpha x\, d\alpha \quad ,$$

where

$$A(\alpha) = \int_0^\infty f(x)\cos\alpha x\, dx$$

Important Definition: Fourier Sine Integral

The **Fourier Integral** of an **odd** function $f(x)$ defined on the real line $(-\infty, \infty)$ is the **sine integral**

$$\frac{2}{\pi}\int_0^\infty B(\alpha)\sin\alpha x\, d\alpha \quad ,$$

where

$$B(\alpha) = \int_0^\infty f(x)\sin\alpha x\, dx$$

Note: If the function $f(x)$ is only defined on the positive real line: $(0, \infty)$, then you can extend it either as an even function, and get a **Fourier cosine integral**, or an as an odd function, and get a **Fourier sine integral**. Both of them are correct.

Problem 20.2: Find the cosine and sine integral representation of the function $f(x) = xe^{-3x}, x > 0$.

Solution:

$$A(\alpha) = \int_0^\infty xe^{-3x}\cos\alpha x\, dx \quad .$$

Using Maple, or a table of intgerals (or if you have half an hour to spare, integration by parts), we have

$$A(\alpha) = \frac{9-\alpha^2}{(\alpha^2+9)^2} \quad .$$

Similarly,

$$B(\alpha) = \int_0^\infty x e^{-3x} \sin \alpha x \, dx$$

Using Maple, or a table of integrals, we have

$$B(\alpha) = \frac{6\alpha}{(\alpha^2+9)^2} \quad .$$

Putting it into the cosine and sine integrals, we have

Fourier-cosine representation:

$$\frac{2}{\pi} \int_0^\infty \frac{(9-\alpha^2)\cos \alpha x}{(\alpha^2+9)^2} \, d\alpha \quad ,$$

Fourier-sine representation:

$$\frac{2}{\pi} \int_0^\infty \frac{6\alpha}{(\alpha^2+9)^2} \sin \alpha x \, d\alpha = \frac{12}{\pi} \int_0^\infty \frac{\alpha}{(\alpha^2+9)^2} \sin \alpha x \, d\alpha \quad .$$

Ans. to 20.2: The Fourier-cosine representation of $f(x)$ is $\frac{2}{\pi} \int_0^\infty \frac{(9-\alpha^2)\cos \alpha x}{(\alpha^2+9)^2} \, d\alpha$ and the Fourier-sine representation is $\frac{12}{\pi} \int_0^\infty \frac{\alpha \sin \alpha x}{(\alpha^2+9)^2} \, d\alpha$.