Dr. Z.'s Calc5 Lecture 2 Handout: The Inverse Laplace Transform and Derivatives

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Theory: The *Laplace Transform* is a **dictionary** that goes from functions of t (usually time) to functions of s. It is often necessary to be able to translate back! Taking the little table of Lecture 1, and reversing it we get:

$$(a) \quad \mathcal{L}^{-1}\{\frac{1}{s}\} = 1$$

$$(b) \quad \mathcal{L}^{-1}\{\frac{1}{s^k}\} = \frac{t^{k-1}}{(k-1)!} \quad (k = 1, 2, 3, ...),$$

$$(c) \quad \mathcal{L}^{-1}\{\frac{1}{s-a}\} = e^{at} \quad ,$$

$$(d) \quad \mathcal{L}^{-1}\{\frac{1}{s^2+k^2}\} = \frac{\sin kt}{k} \quad ,$$

$$(e) \quad \mathcal{L}^{-1}\{\frac{s}{s^2+k^2}\} = \cos kt \quad ,$$

$$(f) \quad \mathcal{L}^{-1}\{\frac{1}{s^2-k^2}\} = \frac{\sinh kt}{k} \quad ,$$

$$(g) \quad \mathcal{L}^{-1}\{\frac{s}{s^2-k^2}\} = \cosh kt \quad .$$

Note that except for (a), each of these formulas contains *infinitely* many facts, since they involve parameters. For example thanks to (b) we know the inverse-Laplace-Transform of $1/s$, $1/s^2$, $1/s^3$ etc.

Problem 2.1: Find $\mathcal{L}^{-1}\{\frac{3}{s} + \frac{5}{s-5}\}$

Solution: By linearity:

$$\mathcal{L}^{-1}\{\frac{3}{s} + \frac{5}{s-5}\} = 3\mathcal{L}^{-1}\{\frac{1}{s}\} + 5\mathcal{L}^{-1}\{\frac{1}{s-5}\} \quad .$$

By (a) (or (b) with $k = 1$)

$$\mathcal{L}^{-1}\{\frac{1}{s}\} = 1 \quad ,$$

By (c)

$$\mathcal{L}^{-1}\{\frac{1}{s-5}\} = e^{5t} \quad .$$

Combining, we have

$$\mathcal{L}^{-1}\{\frac{3}{s} + \frac{5}{s-5}\} = 3\mathcal{L}^{-1}\{\frac{1}{s}\} + 5\mathcal{L}^{-1}\{\frac{1}{s-5}\} = 3 \cdot 1 + 5e^{5t} = 3 + 5e^{5t} \quad .$$

Ans. to 2.1: $\mathcal{L}^{-1}\{\frac{3}{s} + \frac{5}{s-5}\} = 3 + 5e^{5t}$.

Problem 2.2: Find $\mathcal{L}^{-1}\{\frac{2s+1}{s^2+9}\}$

Sol.: Remember, we can always break-up the numerator!

$$\mathcal{L}^{-1}\{\frac{2s+1}{s^2+9}\} = \mathcal{L}^{-1}\{\frac{2s}{s^2+9}\} + \mathcal{L}^{-1}\{\frac{1}{s^2+9}\} = 2\mathcal{L}^{-1}\{\frac{s}{s^2+9}\} + \mathcal{L}^{-1}\{\frac{1}{s^2+9}\} \quad .$$

By (e) and (d) with $k = 3$, this equals

$$2\cos 3t + \frac{1}{3}\sin 3t \quad .$$

Ans. to 2.2: $\mathcal{L}^{-1}\{\frac{2s+1}{s^2+9}\} = 2\cos 3t + \frac{1}{3}\sin 3t$.

When we get **complicated** rational functions of s, we need to do a **partial fraction** decomposition.

Problem 2.3: Evaluate

$$\mathcal{L}^{-1}\{\frac{3s-4}{s^2-3s+2}\}$$

Sol.: First we must **factorize** the denominator (unless it is already factored)

$$\frac{3s-4}{s^2-3s+2} = \frac{3s-4}{(s-1)(s-2)}$$

Now are are looking for **magic** numbers, let's call them A and B, such that

$$\frac{3s-4}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2} \quad .$$

Adding the fractions on the right (using $a/b + c/d = (ad+bc)/bd$):

$$\frac{3s-4}{(s-1)(s-2)} = \frac{A(s-2)+B(s-1)}{(s-1)(s-2)} \quad .$$

The denominators automatically match, but to make this come true, we need the numerators to match:

$$3s-4 = A(s-2)+B(s-1) \quad .$$

Now we plug-in *convenient values*. When $s = 2$ we get

$$3\cdot 2 - 4 = A(2-2)+B(2-1) = 0 + B = B \quad .$$

So $B = 2$. When $s = 1$ we get:

$$3\cdot 1 - 4 = A(1-2)+B(1-1) \quad .$$

So

$$-1 = -A$$

and so $A = 1$. Once we know what A and B are we go back and write:

$$\frac{3s-4}{(s-1)(s-2)} = \frac{1}{s-1} + \frac{2}{s-2} \quad .$$

So far it was just algebra. Only now are we ready to apply $\mathcal{L}^{-1}$.

$$\mathcal{L}^{-1}\{\frac{3s-4}{s^2-3s+2}\} = \mathcal{L}^{-1}\{\frac{1}{s-1}\} + \mathcal{L}^{-1}\{\frac{2}{s-2}\} = \mathcal{L}^{-1}\{\frac{1}{s-1}\} + 2\mathcal{L}^{-1}\{\frac{1}{s-2}\} \quad .$$

Using (c) with $a = 1$ and $a = 2$, we get:

$$e^t + 2e^{2t} \quad .$$

Ans. to 2.3:

$$\mathcal{L}^{-1}\{\frac{3s-4}{s^2-3s+2}\} = e^t + 2e^{2t} \quad .$$

More Theory: The beauty of $\mathcal{L}$ is that it turns *differential* equations into *algebraic* equations!

If $y(t)$ is *any* function of time, and if $Y(s)$ is its Laplace Transform:

$$\mathcal{L}\{y(t)\} = Y(s) \quad ,$$

then the Laplace Transform of the first derivative, $y'(t)$, is "almost" $Y(s)$ multiplied by s, and if $y(0) = 0$ then it is exactly that. We have:

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$$

(Prove this!, Hint: Integration by parts.) Applying this very same formula to $y''(t)$ we have

$$\mathcal{L}\{y''(t)\} = s(sY(s) - y(0)) - y'(0) = s^2Y(s) - sy(0) - y'(0) \quad .$$

One more time:

$$\mathcal{L}\{y'''(t)\} = s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \quad ,$$

and so on. In general:

$$\mathcal{L}\{y^{(n)}(t)\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - y^{(n-1)}(0) \quad .$$

Equipped with this, we can solve **Initial Value Problems**.

Problem 2.4: Use Laplace Transform to solve the following initial-value problem.

$$y' + 2y = e^t \quad , \quad y(0) = -3 \quad .$$

Solution: Let $Y(s) = \mathcal{L}\{y(t)\}$. Applying $\mathcal{L}$:

$$\mathcal{L}\{y' + 2y\} = \mathcal{L}\{e^t\} \quad .$$

So

$$\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^t\} \quad .$$

But $\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) + 3$. Also $\mathcal{L}\{e^t\} = \frac{1}{s-1}$. So we have the **algebraic equation**: (Let's abbrevaiate, and write Y for $Y(s)$)

$$sY + 3 + 2Y = \frac{1}{s-1} \quad .$$

Solving for Y, we get,

$$(s+2)Y = \frac{1}{s-1} - 3 = \frac{4-3s}{s-1} \quad .$$

Dividing both sides by $(s+2)$, we have an *explicit* expression for Y:

$$Y(s) = \frac{4-3s}{(s-1)(s+2)} \quad .$$

In order to find $y(t)$ we need to compute

$$\mathcal{L}^{-1}\{\frac{4-3s}{(s-1)(s+2)}\} \quad .$$

Doing partial fractions:

$$Y(s) = \frac{4-3s}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2} \quad .$$

We get

$$\frac{4-3s}{(s-1)(s+2)} = \frac{A(s+2)+B(s-1)}{(s-1)(s+2)} \quad ,$$

so

$$4 - 3s = A(s+2) + B(s-1) \quad .$$

When $s = 1$, we get $1 = 3A$ so $A = \frac{1}{3}$ and when $s = -2$ we get $10 = B(-3)$ so $B = -\frac{10}{3}$, so the simplified expression for Y is:

$$Y(s) = \frac{\frac{1}{3}}{s-1} + \frac{-\frac{10}{3}}{s+2} \quad .$$

Now we are ready to take $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\{\frac{\frac{1}{3}}{s-1} + \frac{-\frac{10}{3}}{s+2}\} = (\frac{1}{3})\mathcal{L}^{-1}\{\frac{1}{s-1}\} + (-\frac{10}{3})\mathcal{L}^{-1}\{\frac{1}{s+2}\} =$$

$$\frac{1}{3}e^t - \frac{10}{3}e^{-2t} \quad .$$

Ans. to 2.4: $y(t) = \frac{1}{3}e^t - \frac{10}{3}e^{-2t}$.

Problem 2.5: Use Laplace Transform to solve the following initial-value problem.

$$y'' - 3y' + 2y = e^{3t} \quad , \quad y(0) = 0 \quad , \quad y'(0) = 0 \quad .$$

Solution: As usual $\mathcal{L}\{y(t)\} = Y(s)$. Applying $\mathcal{L}$:

$$\mathcal{L}\{y'' - 3y' + 2y\} = \mathcal{L}\{e^{3t}\} \quad , \quad y(0) = 0 \quad , \quad y'(0) = 0 \quad .$$

$$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \frac{1}{s-3}$$

Note that $\mathcal{L}\{y'\} = sY - y(0) = sY$, $\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0) = s^2Y$, so

$$s^2Y - 3sY + 2Y = \frac{1}{s-3} \quad .$$

$$(s^2 - 3s + 2)Y = \frac{1}{s-3} \quad .$$

$$(s-1)(s-2)Y = \frac{1}{s-3} \quad .$$

Solving for Y:

$$Y = \frac{1}{(s-1)(s-2)(s-3)} \quad .$$

Partial Fractions:

$$\frac{1}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3} \quad .$$

$$1 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2) \quad .$$

Convenient values: $s = 1$: $1 = A(-1)(-2)$, so $A = \frac{1}{2}$; $s = 2$: $1 = B(1)(-1)$, so $B = -1$; $s = 3$: $1 = C(2)(1)$, so $C = \frac{1}{2}$. So

$$Y = \frac{\frac{1}{2}}{s-1} + \frac{-1}{s-2} + \frac{\frac{1}{2}}{s-3} \quad .$$

Finally,

$$y(t) = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\{\frac{\frac{1}{2}}{s-1} + \frac{-1}{s-2} + \frac{\frac{1}{2}}{s-3}\} = \mathcal{L}^{-1}\{\frac{1}{2}\frac{1}{s-1} - \frac{1}{s-2} + \frac{1}{2}\frac{1}{s-3}\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\{\frac{1}{s-1}\} - \mathcal{L}^{-1}\{\frac{1}{s-2}\} + \frac{1}{2}\mathcal{L}^{-1}\{\frac{1}{s-3}\} = \frac{1}{2}e^t - e^{2t} + \frac{1}{2}e^{3t} \quad .$$

Ans. to 2.5: $y(t) = \frac{1}{2}e^t - e^{2t} + \frac{1}{2}e^{3t}$.