Dr. Z.'s Calc5 Lecture 12 Handout: Fourier-Legendre Series and Linear Recurrences

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Important Definition

The Legendre polynomials $\{P_n(x)\}_{n=0}^{\infty}$ are defined by the **generating function**

$$\sum_{n=0}^{\infty} P_n(x)t^n = (1-2xt+t^2)^{-1/2} \quad .$$

Another way to define them is via the **recurrence**

$$P_n(x) = \frac{2n-1}{n}xP_{n-1}(x) - \frac{n-1}{n}P_{n-2}(x) \quad ,$$

subject to the **initial values**:

$$P_0(x) = 1 \quad P_1(x) = x \quad .$$

Important Formula

The **Fourier-Legendre series** of a function $f(x)$ defined on the interval $(-1,1)$ is given by

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x) \quad ,$$

where

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx \quad .$$

Problem 12.1: Use the recurrence

$$P_n(x) = \frac{2n-1}{n}xP_{n-1}(x) - \frac{n-1}{n}P_{n-2}(x) \quad ,$$

and $P_0(x) = 1, P_1(x) = x$ to find $P_2(x)$,and $P_3(x)$.

Solution: When $n = 2$, we get

$$P_2(x) = \frac{2\cdot 2-1}{2}xP_{2-1}(x) - \frac{2-1}{2}P_{2-2}(x) = \frac{3}{2}xP_1(x) - \frac{1}{2}P_0(x) = \frac{3}{2}x^2 - \frac{1}{2} \quad .$$

When $n = 3$, we get

$$P_3(x) = \frac{2\cdot 3-1}{3}xP_{3-1}(x) - \frac{3-1}{3}P_{3-2}(x) = \frac{5}{3}xP_2(x) - \frac{2}{3}P_1(x) = \frac{5}{3}x\left(\frac{3}{2}x^2 - \frac{1}{2}\right) - \frac{2}{3}x = \frac{5}{2}x^3 - \frac{5}{6}x - \frac{2}{3}x = \frac{5}{2}x^3 - \frac{3}{2}x \quad .$$

Problem 12.2: Find the first three coefficients of the Fourier-Legendre expansion of

$$f(x) = \begin{cases} 2, & \text{if } -1 < x < 0; \\ 3, & \text{if } 0 \le x < 1. \end{cases}$$

Solution:

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x)\, dx \quad ,$$

where

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx \quad .$$

We only need to find out c_0, c_1, c_2.

$$c_0 = \frac{2\cdot 0+1}{2} \int_{-1}^{1} f(x) P_0(x)\, dx = \frac{1}{2} \int_{-1}^{1} f(x)\, dx = \frac{1}{2} \int_{-1}^{0} f(x)\, dx + \frac{1}{2} \int_{0}^{1} f(x)\, dx =$$

$$\frac{1}{2} \int_{-1}^{0} 2\, dx + \frac{1}{2} \int_{0}^{1} 3\, dx = \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 3 = \frac{5}{2} \quad .$$

$$c_1 = \frac{2\cdot 1+1}{2} \int_{-1}^{1} f(x) P_1(x)\, dx = \frac{3}{2} \int_{-1}^{1} f(x) x\, dx = \frac{3}{2} \int_{-1}^{0} f(x) x\, dx + \frac{3}{2} \int_{0}^{1} f(x) x\, dx =$$

$$\frac{3}{2} \int_{-1}^{0} 2x\, dx + \frac{3}{2} \int_{0}^{1} 3x\, dx = \frac{3}{2}\cdot(0-1) + \frac{3}{2}\cdot\frac{3}{2} = \frac{3}{4} \quad .$$

$$c_2 = \frac{2\cdot 2+1}{2} \int_{-1}^{1} f(x) P_2(x)\, dx = \frac{5}{2} \int_{-1}^{1} f(x)(\frac{3}{2}x^2 - \frac{1}{2})\, dx = \frac{5}{2} \int_{-1}^{0} f(x)(\frac{3}{2}x^2 - \frac{1}{2})\, dx + \frac{5}{2} \int_{0}^{1} f(x)(\frac{3}{2}x^2 - \frac{1}{2})\, dx =$$

$$= \frac{5}{2} \int_{-1}^{0} 2(\frac{3}{2}x^2 - \frac{1}{2})\, dx + \frac{5}{2} \int_{0}^{1} 3(\frac{3}{2}x^2 - \frac{1}{2})\, dx = \frac{5}{2} \int_{-1}^{0} (3x^2 - 1)\, dx + \frac{5}{2} \int_{0}^{1} (\frac{9}{2}x^2 - \frac{3}{2})\, dx$$

$$= \frac{5}{2}(x^3 - x)\Big|_{-1}^{0} + \frac{5}{2}(\frac{3}{2}x^3 - \frac{3}{2}x)\Big|_{0}^{1} = \frac{5}{2}((0^3 - 0) - ((-1)^3 - (-1)) + \frac{5}{2}(0 - 0) = 0 \quad .$$

Ans. to 12.2: $c_0 = \frac{5}{2}$, $c_1 = \frac{3}{4}$, $c_2 = 0$.