Dr. Z.'s Calc5 Lecture 11 Handout: Sturm-Liouville Problem

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Calc4 Reminders

The general solution of

$$y' + \alpha y = 0 \quad is \quad y = c_1 e^{-\alpha x} \quad .$$

$$y'' + \alpha^2 y = 0 \quad is \quad y = c_1 \cos \alpha x + c_2 \sin \alpha x \quad .$$

Important Definition

A **Regular Sturm-Liouville Problem** on an interval $[a, b]$ is a **differential equation** of the form

$$\frac{d}{dx}[r(x)y'] + (q(x) + \lambda p(x))y = 0 \quad ,$$

Subject to the **boundary conditions**

$$A_1 y(a) + B_1 y'(a) = 0$$

$$A_2 y(b) + B_2 y'(b) = 0$$

Here p, q, r are continuous functions, and in addition $r'(x)$ should also be continuous. Also we need $r(x) > 0$ and $p(x) > 0$ on the interval $[a, b]$.

Singular Sturm-Liouville Problem on an interval $[a, b]$ is a **differential equation** of the above form but the condition that $r(x) > 0$ in $[a, b]$ is not always true, but then you only use some of the boundary conditions.

For most λ's there is **no solution** (except for the "trivial solution" $y(x) = 0$). Those lucky ones for which there is a non-zero solution are called **eigenvalues** and the corresponding solutions are called **eigenfunctions**.

Important Theorem

1. For a regular Sturm-Liouville problem there exist an infinite number of eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots$$

such that $\lambda_n \to \infty$.

2. Each **eigenvalue** λ_i has just one corresponding eigenfunction $y_i(x)$ (up to a constant multiple)

3. All the eigenfunctions are **linearly independent**. In other words, there is no way that you can express one of them as a linear combination of other ones.

4. The eigenvectors $\{y_i(x)\}$ are **orthogonal** over $[a, b]$ with respect to the **weight-function** $p(x)$.

Problem 11.1: Find the eigenfunctions and the equation that defines the eigenvalues for the following boundary value problem.

$$y'' + \lambda y = 0 \quad , \quad y'(0) = 0 \quad , \quad y(\pi) + y'(\pi) = 0 \quad .$$

Solution of 11.1: It is more convenient to replace λ by λ^2, and consider:

$$y'' + \lambda^2 y = 0 \quad , \quad y'(0) = 0 \quad , \quad y(\pi) + y'(\pi) = 0 \quad .$$

The **general solution** of the DE is

$$y(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x) \quad ,$$

for **some** *constants* to be determined. First we find the **derivative**

$$y'(x) = -c_1 \lambda \sin(\lambda x) + c_2 \lambda \cos(\lambda x) \quad .$$

Now we use the **boundary conditions**. Plugging in $x = 0$ in $y'(x)$ gives

$$y'(0) = -c_1 \lambda \sin(\lambda \cdot 0) + c_2 \lambda \cos(\lambda \cdot 0) = -c_1 \lambda \sin(0) + c_2 \lambda \cos(0) = -c_1 \cdot \lambda \cdot 0 + c_2 \cdot \lambda \cdot 1 = c_2 \lambda \quad .$$

The first boundary condition tells you that $y'(0) = 0$, so we **must** have $c_2 = 0$ (since $\lambda \neq 0$). So, it is easier to write

$$y(x) = c_1 \cos(\lambda x) \quad .$$

and so

$$y'(x) = -c_1 \lambda \sin(\lambda x) \quad .$$

And

$$y(x) + y'(x) = c_1 \cos(\lambda x) - c_1 \lambda \sin(\lambda x) = c_1(\cos(\lambda x) - \lambda \sin(\lambda x))$$

Plugging-in $x = \pi$ gives

$$y(\pi) + y'(\pi) = c_1(\cos(\lambda \pi) - \lambda \sin(\lambda \pi))$$

Since $c_1 \neq 0$ (always), we get the **eigenvalue condition**:

$$\cos(\lambda \pi) - \lambda \sin(\lambda \pi) = 0 \quad ,$$

which is the same as (divide by $\sin(\lambda \pi)$)

$$\cot(\lambda \pi) - \lambda = 0 \quad .$$

So the eigenvalues are the *infinitely many* roots of this equation, that a computer can approximate using Newton's rule. Calling these numbers $\lambda_1, \lambda_2, \ldots$, the corresponding **eigenfunctions** are $\{\cos \lambda_n x\}$, $n = 1, 2, 3, \ldots$.

Ans. to 11.11: The eigenvalues are the infinite sequence of roots of the "transcendental equation" (that only computers can solve, and only approximately!) $\lambda = \cot(\pi\lambda)$, and calling these numbers λ_n, the eigenfunctions are $\{\cos \lambda_n x\}$, $n = 1, 2, 3, \ldots$.