Dr. Z.'s Calc5 Lecture 10 Handout: Complex Fourier Series

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Important Definition (Fundamental Interval): The **complex Fourier series** of a function f defined on an interval $(-\pi, \pi)$ is given by

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \quad ,$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx \quad , \quad n = 0, \pm 1, \pm 2, \ldots \quad .$$

Important Definition (General Interval): The **complex Fourier series** of a function f defined on an interval $(-p, p)$ is given by

$$\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p} \quad ,$$

where

$$c_n = \frac{1}{2p} \int_{-p}^{p} f(x) e^{-in\pi x/p}\, dx \quad , \quad n = 0, \pm 1, \pm 2, \ldots \quad .$$

If f is a "nice" function then the Fourier series converges to $f(x)$ wherever $f(x)$ is continuous, and to the average of the limits from the left and from the right at the points of discontinuity.

Problem 10.1: Expand $f(x) = x^2$, $-\pi < x < \pi$, in a complex Fourier series.

Solution:

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 e^{-inx}\, dx \quad .$$

When $n = 0$, this is easy:

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2\, dx = \frac{1}{2\pi} \left(\frac{x^3}{3} \Big|_{-\pi}^{\pi} \right) = \frac{1}{2\pi} \frac{2}{3} \pi^3 = \frac{\pi^2}{3}$$

Remember (integration by parts, or use Maple)

$$\int x^2 e^{cx}\, dx = \left(\frac{2}{c^3} - \frac{2x}{c^2} + \frac{x^2}{c} \right) e^{cx}$$

Here $c = -in$ $(n \neq 0)$ so:

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 e^{-inx}\, dx = \frac{1}{2\pi} \left(\frac{2}{in^3} - \frac{2x}{-n^2} + \frac{x^2}{-in} \right) e^{-inx} \Big|_{-\pi}^{\pi} =$$

$$\frac{1}{2\pi}\left(\frac{2}{in^3}-\frac{2\pi}{-n^2}+\frac{\pi^2}{-in}\right)e^{-in\pi}-\frac{1}{2\pi}\left(\frac{2}{in^3}-\frac{-2\pi}{-n^2}+\frac{\pi^2}{-in}\right)e^{in\pi}$$

Now $e^{i\pi}=-1$, so $e^{in\pi}=(-1)^n$. and

$$c_n=\frac{1}{2\pi}\left(\frac{2}{in^3}-\frac{2\pi}{-n^2}+\frac{\pi^2}{-in}\right)(-1)^n-\frac{1}{2\pi}\left(\frac{2}{in^3}-\frac{-2\pi}{-n^2}+\frac{\pi^2}{-in}\right)(-1)^n$$

$$=\frac{1}{2\pi}\left(-2\frac{2\pi}{-n^2}\right)(-1)^n$$

$$=\frac{2(-1)^n}{n^2}$$

So, in $(-\pi,\pi)$, we have:

$$x^2=\frac{\pi^2}{3}+\sum_{n=-\infty,n\neq 0}^{\infty}\frac{2(-1)^n}{n^2}e^{inx} \quad .$$

Ans. to 10.1: $x^2=\frac{\pi^2}{3}+\sum_{n=-\infty,n\neq 0}^{\infty}\frac{2(-1)^n}{n^2}e^{inx}$.

Important Definition: Frequency Spectrum

If a function f(x) is defined on the interval $(-p,p)$, its *periodic extension* is obtained by translating that same function to $(p,3p),(3p,5p),(5p,7p)$, etc. to the right and $(-3p,-p),(-5p,-3p)$ etc. to the left. Its **fundamental period**, T, is simply $2p$. Its **fundamental angular frequency**, ω, is simply $2\pi/T=\pi/p$. For the "default" case of $\sin x$ and $\cos x$, $T=2\pi$ and $\omega=1$.

The **frequency spectrum** is the plot of the infinite set of points

$$(n\omega,|c_n|),n=0,\pm1,\pm2,\ldots.$$

Problem 10.2: Using 10.1, find the frequency spectrum of the periodic extension of $f(x)=x^2$ defined on $(-\pi,\pi)$.

Solution: $\omega=2\pi/2\pi=1$ so

$$(0,\frac{\pi^2}{3}) \quad , \quad (n\omega,|c_n|)=(n,\frac{2}{n^2}),n=\pm1,\pm2,\ldots.$$

Ans. to 10.2: $(0,\frac{\pi^2}{3})$, $(n,\frac{2}{n^2}),n=\pm1,\pm2,\ldots$.