Dr. Z.'s Shortcut Methods for Solving Boundary Value Problems for PDEs

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Fourier Series (over $(-\pi, \pi)$)

Every function defined on the interval $(-\pi, \pi)$ can be written as a *finite* or (more often infinite) **linear combination** of pure sine-waves and pure cosine-waves (and the constant function).

If it ain't broke don't fix it

If the function is given as either a pure sine-wave ($\sin nx$ for some **integer** n), or pure cosine-wave ($\cos nx$ for some **integer** n), or is a constant function (e.g. 8), then: **Its Fourier Series is Itself!**.

Also if it is a *finite* combination of pure sine and/or cosine waves.

Examples: The Fourier series over $(-\pi, \pi)$ of the following functions are themselves.

$$f(x) = 5 \quad , \quad f(x) = 11\cos 7x \quad , \quad f(x) = 11\sin 3x \quad , \quad f(x) = -6\sin x + 10 + 11\sin 3x - \cos 5x + 3\cos 8x \quad .$$

Non-Examples: The Fourier series over $(-\pi, \pi)$ of the following functions are **NOT** themselves

$$f(x) = 5\sin(x/2) \quad , \quad f(x) = \cos(7x/3) \quad , \quad f(x) = x \quad , \quad f(x) = x^2 \quad .$$

Only if the function $f(x)$ is not a pure sine-waves or cosine-waves or a finite linear combination of these, do you have to use the **formula**.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad ,$$

where the number a_0 is given

$$a_0 := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx \quad ,$$

and the numbers $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are given by:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \quad ,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \quad .$$

Fourier Series (over $(-L, L)$) find the function $g(x) = f(xL/\pi)$, that is defined over $(-\pi, \pi)$, and then go back using $f(x) = g(x\pi/L)$.

In this case the building blocks are $\sin((\pi/L)nx)$, $\cos((\pi/L)nx)$ and the constant functions. So if you have to find the Fourier series of $3\sin((5/2)x) + 11\cos((11/2)x)$ over $(-2\pi, 2\pi)$, it would be the same as the function. Even $\sin(5x)$ would be OK, since it has the right format $\sin((10/2)x)$. On the other hand $\sin((11/4)x)$ would not be its own Fourier series, you have to do it the long way.

Half Range Fourier Cosine Series

The Fourier Cosine series of a function $f(x)$ defined on the interval $(0, \pi)$ is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad ,$$

where

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx \quad ,$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx \quad .$$

But if the given function is a combination of $\cos nx$ (n integer) then its Fourier-Cosine series equals itself. For example the Half-Range Fourier-Cosine Series of $f(x) = 5 + 2\cos 4x - 11\cos 7x$ equals itself! On the other hand if $f(x) = \sin x$ or $f(x) = \cos(7x/2)$ you would have to do it the long way, using the formulas.

Half Range Fourier Sine Series

The Fourier Sine series of a function $f(x)$ defined on the interval $(0, \pi)$ is:

$$\sum_{n=1}^{\infty} b_n \sin nx \quad ,$$

where

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\, dx \quad .$$

But if the given function is a combination of $\sin nx$ (n integer) then its Half-Range Fourier-Sine series equals itself. For example, the Half-Range Fourier-Sine Series of $f(x) = 2\sin 4x - 11\sin 17x$ equals itself! On the other hand if $f(x) = \cos x$ and even if $f(x) = 1$, you would have to do it the long way, using the formulas.

Dr. Z.'s Way of Solving the Heat Equation

1. Both ends are at temperature 0: (General interval $(0, L)$)

The solution of

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad , \quad 0 < x < L \quad , \quad t > 0$$

subject to

$$u(0,t)=0 \quad , \quad u(L,t)=0 \quad , \quad t>0$$

$$u(x,0)=f(x) \quad , \quad 0<x<L \quad .$$

Instead of using the stupid formula, remember that the **building block** solutions are $u(x,t) = e^{-k(n^2\pi^2/L^2)t}\sin\frac{n\pi}{L}x$. For this function, $u(x,0)=\sin\frac{n\pi}{L}x$ so if you are lucky and the initial condition function $f(x)$ is a multiple of $\sin\frac{n\pi}{L}x$, for some specific *integer* n, then all you have to do, to get the solution $u(x,t)$ is to **stick** $e^{-k(n^2\pi^2/L^2)t}$ in front of it! If it is a combination of $\sin\frac{n\pi}{L}x$ for various n's just stick the appropriate $e^{-k(n^2\pi^2/L^2)t}$ (for the appropriate n) to each term.

Example: Solve the pde

$$5\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t} \quad , \quad 0<x<\pi \quad , \quad t>0$$

subject to

$$u(0,t)=0 \quad , \quad u(\pi,t)=0 \quad , \quad t>0 \quad ,$$

$$u(x,0)=5\sin(3x)-8\sin(7x) \quad , \quad 0<x<\pi \quad .$$

Sol. Here $k=5$, we first **copy-and-paste** $f(x)$, and leave some room, as follows:

$$u(x,t)=5(ComingUpShortly1)\sin(3x)-8(ComingUpShortly2)\sin(7x) \qquad (NotYetFinished)$$

$ComingUpShortly1$ is simply $e^{-k(n^2\pi^2/L^2)t}$ with $k=5$, $n=3$ and $L=\pi$, i.e.

$$ComingUpShortly1=e^{-5(3^2(\pi^2/\pi^2)t}=e^{-45t} \quad .$$

Similarly $ComingUpShortly2$ is simply $e^{-k(n^2\pi^2/L^2)t}$ with $k=5$, $n=7$ and $L=\pi$, i.e.

$$ComingUpShortly2=e^{-5(7^2(\pi^2/\pi^2)t)}=e^{-245t} \quad .$$

Going back to $(NotYetFinished)$

$$u(x,t)=5e^{-45t}\sin(3x)-8e^{-245t}\sin(7x) \quad . \qquad (Finished)$$

If however, the $f(x)$ of the problem is not a pure sine-wave or a finite combination of them, for example $u(x,0)=x$ or $u(x,0)=\cos x$, then you have to find the Half-Range Fourier Sine Expansion, as above, get a $\sum$,

$$u(x,0)=f(x)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi}{L}x \quad ,$$

and do exactly the same procedure as above! Stick $e^{-k(n^2\pi^2/L^2)t}$ between A_n and $\sin\frac{n\pi}{L}x$, to get the answer:

$$u(x,t)=\sum_{n=1}^{\infty}A_n e^{-k(n^2\pi^2/L^2)t}\sin\frac{n\pi}{L}x \quad .$$

Note that *now* n is a general symbol, so you leave it alone! You only plug-in the numerical values of L (often $L=\pi$, the easiest case), and k.

2. Both ends are insulated

Things are exactly analogous, but now you use the **Fourier-Cosine** Half-Range expansion, and stick the $e^{-k(n^2\pi^2/L^2)t}$ between A_n and $\cos\frac{n\pi}{L}x$.

(Note, in many problems things simplify since $L=\pi$). Of course if the initial-condition function is already a combination of pure-cosines, you leave it alone, and do the "sticking" as above.

Wave Equation (Special case: $L=\pi$)

To find the solution of the boundary value wave equation

$$a^2u_{xx}=u_{tt} \quad , \quad 0<x<\pi \quad , \quad t>0 \quad ;$$

$$u(0,t)=0 \quad , \quad u(\pi,t)=0 \quad , \quad t>0 \quad ;$$

$$u(x,0)=f(x) \quad , \quad u_t(x,0)=g(x) \quad , \quad 0<x<\pi \quad .$$

Step 1. Find the Fourier Sine Expansion of $f(x)$ and the Fourier Sine Expansion of $g(x)$, writing

$$f(x)=\sum_{n=1}^{\infty}a_n\sin nx \quad ,$$

$$g(x)=\sum_{n=1}^{\infty}b_n\sin nx \quad .$$

For some numbers a_n and b_n (or expressions in n).

Important note: If $f(x)$ and $g(x)$ are already in that format, but there are only finitely many terms, leave them alone, you don't have to do anything!

To get the answer $u(x,t)$ you first write, tentatively

$$u(x,t)=\sum_{n=1}^{\infty}a_n(ComingUpShortly1)\sin nx+\sum_{n=1}^{\infty}b_n(ComingUpShortly2)\sin nx \quad , \qquad (NotYetDone)$$

Now for each n,

$$ComingUpShortly1=\cos(nat) \quad ,$$

$$ComingUpShortly2 = \frac{\sin(nat)}{na} \quad .$$

If you are lucky, and both $f(x)$ and $g(x)$ are finite combinations of pure sine-waves (or a single sine-wave), then you do it to the finite expression. **Much faster than blindly following formulas**.

Example: Find the solution of the boundary value wave equation

$$36u_{xx} = u_{tt} \quad , \quad 0 < x < \pi \quad , \quad t > 0 \quad ;$$

$$u(0,t) = 0 \quad , \quad u(\pi, t) = 0 \quad , \quad t > 0 \quad ;$$

$$u(x, 0) = \sin 3x \quad , \quad u_t(x, 0) = 2\sin 4x + 6\sin 7x \quad , \quad 0 < x < \pi \quad .$$

Sol. Here $a = 6$.

$$u(x,t) = (ComingUpShortly1)\sin 3x + 2(CmoingUpShortly2a)\sin 4x + 6(CmoingUpShortly2b)\sin 7x. \qquad (NotYetDone)$$

$$ComingUpShortly1 = \cos(3 \cdot 6t) = \cos 18t$$

(since now $n = 3$ and $a = 6$.)

$$ComingUpShortly2a = \frac{\sin(4 \cdot 6t)}{4 \cdot 6} = \frac{\sin(24t)}{24} \quad .$$

(since now $n = 4$ and of course $a = 6$.)

$$ComingUpShortly2b = \frac{\sin(7 \cdot 6t)}{7 \cdot 6} = \frac{\sin(42t)}{42} \quad .$$

(since now $n = 7$ and of course $a = 6$.) Going back to $(NotYetDone)$, we get that the answer is:

$$u(x,t) = (\cos 18t)(\sin 3x) + 2(\frac{\sin(24t)}{24})(\sin 4x) + 6(\frac{\sin(42t)}{42})(\sin 7x). \qquad (AlmostDone)$$

Now you just **clean up** to get:

$$u(x,t) = \cos 18t \sin 3x + \frac{\sin 24t \sin 4x}{12} + \frac{\sin(42t)\sin 7x}{7}. \qquad (Done)$$

Laplace's Equation in a Rectangle $u_{xx} + u_{yy} = 0$ (The Hardest Topic in this semester !)

The **catalog** of the **building blocks** obtained **once and for all** from the technique called **separation of variables** are

$$\cos\lambda x \cosh(\lambda y) \quad , \quad \cos\lambda x \sinh(\lambda y) \quad , \quad \sin\lambda x \cosh(\lambda y) \quad , \quad \sin\lambda x \sinh(\lambda y) \quad ,$$

and

$$\cosh\lambda x \cos(\lambda y) \quad , \quad \cosh\lambda x \sin(\lambda y) \quad , \quad \sinh\lambda x \cos(\lambda y) \quad , \quad \sinh\lambda x \sin(\lambda y) \quad .$$

Here λ is *any* real number.

Given a complicated boundary value problem, you use the **boundary superposition principle** to break them up into easier problems, where three of the four sides are set to 0 and only one side is non-zero. Then step-by-step you kick out those functions that do not meet the conditions $u(x,0) = 0$ and $u(0,y) = 0$. Then λ gets narrowed-down to integer n (or some multiple of n if the x-side does not have length π). Then you write down the infinite linear combination for $u(x,y)$, and use the only non-zero boundary condition to plug-in, get some Fourier-Sine or Fourier-Cosine Expansion, as the case may be, and compare it to the function given as the last side's boundary condition. If you are lucky and it is already expessible as a *finite* combination (or just a pure sine- or cosine- wave), then you do the same trick as above. Otherwise, you find the Fourier-Sine or Fourier-Cosine and do analogous things.

Laplace's Equation in a Circle (in Polar) $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$. (A Piece Of Cake!)

To find the steady-state temperature in a circle of radius c where $u(c,\theta) = f(\theta)$.

Step 1:

Find the **full** Fourier series of $f(\theta)$

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta \quad .$$

Warning: If you are lucky and the given function $f(\theta)$ is a a pure sine-wave or a pure cosine-way, or a finite linear combination of these, you do nothing! Leave it alone.

Step 2: Stick $(r/c)^n$ between a_n and $\cos n\theta$ (if applicable) and Stick $(r/c)^n$ between b_n and $\sin n\theta$ (if applicable). That's it! Getting

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n (r/c)^n \cos n\theta + \sum_{n=1}^{\infty} b_n (r/c)^n \sin n\theta \quad .$$

Example of the lucky case: Find the steady-state temperature in a circle of radius 5 if the temperature in the circumference $r = 5$ is given by $u(5,\theta) = 5 + \sin 3\theta - 3\cos 8\theta$.

Sol.

$$u(r,\theta) = 5 + (ComingUpShortly1)\sin 3\theta - 3(ComingUpShortly2)\cos 8\theta \qquad (NotYetDone)$$

$ComingUpShorly1$ is $(r/5)^3$ and $ComingUpShorly2$ is $(r/5)^8$ and the answer is:

$$u(r,\theta) = 5 + (r/5)^3 \sin 3\theta - 3(r/5)^8 \cos 8\theta \quad . \qquad (Done)$$

That's it!

WARNING: That's it! The answer, $u(r, \theta)$, is a function of the *variables* r and θ. r is **NOT** 5, c is 5. Do not "simplify" the answer and plug-in at the end $r = 5$. You would get **no credit**, since this is **nonsense** (or rather you would get $f(\theta)$ back, so it is a good check, but it is not the answer).