Dr. Z.'s Calc5 Last Handout and SCC Problems : The Art of Checking Your Answer and Avoiding Nonsense

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Note: The exercises here are **mandatory** for members of the SCC II. There are strongly recommended to **everyone!**

Avoid Grammatically Incorrect Answers

Everyone makes mistakes (myself included) but if the answer is gibberish, then it means that you don't understand the *language* of Mathematics, and you are just faking it, trying to follow the steps, but not having any clue what it means.

You must understand the **meaning** of notation. We often see the $\sum$ symbol, for example, in Lecture 8 we had the following

Problem 8.1: Find the Fourier series of

$$f(x) = \begin{cases} 1, & \text{if } -\pi < x < 0; \\ -3, & \text{if } 0 \leq x < \pi. \end{cases}$$

and the answer was

$$f(x) \; = \; -1 + \sum_{n=1}^{\infty} \frac{4((-1)^n - 1)}{\pi n} \sin nx \quad .$$

This is just **shorthand** for

$$-1 + \frac{4((-1)^1 - 1)}{\pi(1)} \sin x + \frac{4((-1)^2 - 1)}{\pi 2} \sin 2x + \frac{4((-1)^3 - 1)}{\pi 3} \sin 3x + \frac{4((-1)^4 - 1)}{\pi 4} \sin 4x + \ldots$$

$$= -1 - \frac{8}{\pi} \sin x - \frac{8}{\pi 3} \sin 3x + \ldots \quad .$$

So if we plug-in *any* numerical value into this infinite series we get a convergent series that converges to the value of $f(x)$. For example, if $x = .5$ we should get 1, and if $x = .8$ we should get -3.

In practice, of course, we can't compute *infinitely* many terms, so our computer finds for us the sum of the first 1000 (or whatever) terms, and we get a good *approximation*. Since in science there is always experimental error, this is good enough for all practical purposes.

Do right now!

1. Spell out the first 5 terms in the following $\sum$-s continued by $\ldots$

[**Note added Dec. 11, 2011**: I thank Chris Farina for correcting a typo in the previous version, that started at $n = 0$, and of course the answer to the original question (when the summation started with $n = 0$) was "undefined".]

$$(a) \quad \sum_{n=1}^{\infty} \frac{1}{\pi n^3} \cos nx \quad ,$$

2. Spell out the first 4 terms in the following $\sum$-s continued by ...

$$(b) \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi n^2} \sin nx \; e^{-n^2 t} \quad ,$$

Checking Solutions of Algebraic Equations

Any algebraic equation is a *puzzle.* For example the quadratic equation:

$$x^2 - 3x + 2 = 0$$

asks for a number such that you if you take its square, subtract 3 times itself and add 2, you would get the number 0.

Quadratic equations are easy to solve, but higher degree ones are either hard or impossible. But to *check* that a proposed solution of an algebraic equation is indeed a solution, all you have to do is **plug-it-in**.

For example, to check that $x = 2$ is indeed a solution of the above equation, we replace x by 2 and check whether

$$2^2 - 3(2) + 2 = 0$$

Since $4 - 6 + 2 = 0$, this is true!

Do Right now

3. Check that $x = 2$ is a solution of the algebraic equation $x^5 - x^2 - 28 = 0$.

4. Check that $x = 2, y = 1, z = 3$ is a solution of the system of algebraic equations

$$x + y - z = 0 \quad , \quad x - y + z = 4 \quad , \quad 2x + y + 3z = 14 \quad .$$

Checking Answers to ODEs

An **ordinary** differential equation (**ode** for short) is also a *puzzle*, but now the answers are functions! It may happen that the answer *looks* like a number, for example $y(t) = 4$, but this is an illusion. In this context 4 means the **constant function** that is always equal to 4.

For example, the ode, plus the initial conditions

$$y''(t) - 3y'(t) + 2y(t) = 2t - 3 \quad , \quad y(0) = 2 \quad , \quad y'(0) = 4 \quad .$$

That you are can do either by calc4 methods, or using the Laplace Transform, is a puzzle. In English it means:

"I am a function of t. My second derivative minus three times my first derivative plus myself equal to the function $t-3$. Not only that, when $t=0$, my value happens to be 2 and the value of my derivative when $t=0$ equals 4. Who am I?"

It takes *me* about ten minutes to solve this, and it may take you even longer (but it takes Maple less than a second). The answer turns out to be

$$y(t) = e^t + e^{2t} + t \quad .$$

But to *check* whether this answer is correct, is much faster. **Plug-it in!**.

First we need to find $y'(t)$ and $y''(t)$:

$$y'(t) = e^t + 2e^{2t} + 1 \quad .$$

$$y''(t) = e^t + 4e^{2t} \quad .$$

Plugging these into the ode:

$$e^t + 4e^{2t} - 3(e^t + 2e^{2t} + 1) + 2(e^t + e^{2t} + t) = e^t + 4e^{2t} - 3e^t - 6e^{2t} - 3 + 2e^t + 2e^{2t} + 2t = 2t - 3 \quad .$$

So the ode works out! Now we plug-in the *initial conditions.*

$$y(0) = e^0 + e^{2\cdot 0} + 0 = 1 + 1 + 0 = 2 \quad .$$

$$y'(0) = e^0 + 2e^{2\cdot 0} + 1 = 1 + 2 + 1 = 4 \quad .$$

So I didn't lie, and the proposed solution of the IVP ode above is correct.

Do Right Now

5. Find out whether $y(t) = e^{2t} + e^t$ is a solution of the following IVP ode:

$$y'' - 3y' + 2y = 0 \quad , \quad y(0) = 2 \quad , \quad y'(0) = 3 \quad .$$

6. Find out whether $y(t) = e^{-t} + e^t + e^{2t}$ is a solution of the following IVP ode:

$$y'' - y = 3e^{2t} \quad , \quad y(0) = 1 \quad , \quad y'(0) = 2 \quad .$$

7. Find out whether the functions $x(t) = e^t + e^{2t}$, $y(t) = e^t - e^{2t}$ solve the following *system* of two odes with IVPs.

$$\frac{d^2x}{dt^2} = \frac{5}{2}x - \frac{3}{2}y$$

$$\frac{d^2y}{dt^2} = -\frac{3}{2}x + \frac{5}{2}y$$

$$x(0) = 2 \quad , \quad x'(0) = 3 \quad ; y(0) = 0 \quad , \quad y'(0) = -1 \quad .$$

Checking Answers to PDEs

A *partial differential equation* is also a puzzle, but now you are looking for a *multi-variable* function (in this course, we only focus on functions of two variables, lucky you!).

For example the following Boundary Value Problem PDE

$$u_{xx} + u_{yy} = 0 \quad , \quad (0 < x < 1 \quad , \quad 0 < y < 2) \quad ,$$

$$u(x,0) = x^2 \quad (0 < x < 1) \quad , \quad u(x,2) = x^2 - 4 \quad (0 < x < 1) \quad ,$$

$$u(0,y) = -y^2 \quad (0 < y < 2) \quad , \quad u(1,y) = 1 - y^2 \quad (0 < y < 2) \quad .$$

This is a very hard problem to solve by the techniques we learned in this class. First we need to use the principle of superposition to break it up into four simpler boundary value problems. Then we would need to find which of the **building blocks** (out of the eight options) is applicable in each case, then we would need to find Fourier Sine or Cosine (as the case may be) Expansions for the non-zero boundary function for each case, get the answer to each individual problem, and at the very end, add them up! This i **so complicated** to do by hand, that you won't be asked to solve such a pde. **But** if I tell you to **check** that a proposed solution is indeed a solution, all you have to do it plug-it in.

I claim that $u(x,y) = x^2 - y^2$ is a solution.

Indeed:

$$u_{xx} = 2 \quad , \quad u_{yy} = -2.$$

So $u_{xx} + u_{yy} = 0$, and the pde is OK!

Also

$$u(x,0) = x^2 - 0^2 = x^2 \quad (ok!) \quad , \quad u(x,2) = x^2 - 2^2 = x^2 - 4 \quad (ok!) \quad .$$

$$u(0,y) = 0^2 - y^2 = -y^2 \quad (ok!) \quad , \quad u(1,y) = 1^2 - y^2 = 1 - y^2 \quad (ok!) \quad .$$

Do Right Now

8. Check whether $u(x,y) = x^3y - xy^3 + 2x^2 - 2y^2 + x$ is a solution of the following boundary value problem pde.

$$u_{xx} + u_{yy} = 0 \quad , \quad (0 < x < 1 \quad , \quad 0 < y < 2) \quad ,$$

$$u(x,0) = 2x^2 + x \quad (0 < x < 1) \quad , u(x,2) = 2x^3 + 2x^2 - 7x - 8 \quad (0 < x < 1) \quad ,$$

$$u(0,y) = -2y^2 \quad (0 < y < 2) \quad , u(1,y) = -y^3 - 2y^2 + y + 3 \quad (0 < x < 1) \quad .$$

9. Check whether $u(x,t) = \cos 3t \sin 3x + \frac{1}{4} \sin 4t \sin 4x$ is a solution of the pde

$$u_{xx} = u_{tt} \quad , 0 < x < \pi \quad , \quad t > 0 \quad ;$$

subject to the boundary conditions

$$u(0,t) = 0 \quad , \quad u(\pi,t) = 0 \quad , \quad t > 0 \quad ;$$

and the initial conditions

$$u(x,0) = \sin 3x \quad , \quad u_t(x,0) = \sin 4x \quad , \quad 0 < x < \pi \quad .$$

10. Check whether $u(x,t) = 10 \cos t \sin x + \frac{2}{3} \sin 3t \sin 3x$ is a solution of the pde

$$u_{xx} = u_{tt} \quad , 0 < x < \pi \quad , \quad t > 0 \quad ;$$

subject to the boundary conditions

$$u(0,t) = 0 \quad , \quad u(\pi,t) = 0 \quad , \quad t > 0 \quad ;$$

and the initial conditions

$$u(x,0) = 10 \sin x \quad , \quad u_t(x,0) = 2 \sin 3x \quad , \quad 0 < x < \pi \quad .$$

11. Recall that **Laplace's Equation** in **Polar** coordinates is this

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) u(r,\theta) = 0 \quad .$$

a.. Check that $u(r,\theta) = r^5 \cos 5\theta$ is a solution of this pde.

b.. Check that $u(r,\theta) = r^{-3} \sin 3\theta$ is a solution of this pde.

c.. Check that for every integer n, positive or negative $u(r,\theta) = r^n \sin n\theta$ is a solution of this pde.

d.. Check that for every integer n, positive or negative $u(r,\theta) = r^n \cos n\theta$ is a solution of this pde.

Series and Integral Representations

Most functions are *random* functions, that can only be given numerically, in terms of a table or chart. But some functions can be described most succinctly by an algebraic, trig, or exponential expression. For example, the function $f(x) = x^3$ tells you that in order to find the output for any possible input, all you have to do is raise it to the third power. The function $f(x) = e^{x^3 \sin x} \cos(\sin x^2)$ may look complicated, but it is still better than just a table. For any input, to get the output, all you have to do is plug-it-in.

In applications sometimes functions are given in terms of a *series* representation, the most famous examples, being *Maclaurin* series, for example

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{720} + \dots$$

that gives it in terms of an *infinite* series. In practice (a computer) only uses the first one thousand terms (and often only ten!), and gets a very good approximation (the error is so small that it does not show up in the ten-digit display of the calculator).

This same infinite series for $\sin x$ can also be written in $\sum$-notation

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^n \quad ,$$

and computers and calculators replace the ∞ by 1000 or whatever.

In the case of Maclaurin series the "atomic" functions are the powers $x, x^2, x^3 \dots$.

In the case of Fourier-Series representation, the atomic functions are $\sin nx$ and $\cos nx$, for functions defined on $(-\pi, \pi)$. For the general symmetric interval $(-L, L)$, we have the more complicated-looking "atoms":

$$\sin(n\frac{\pi}{L}x) \quad , \quad \cos(n\frac{\pi}{L}x) \quad .$$

The general format is, for the simpler case of $(-\pi, \pi)$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \cos nx$$

For some *numbers* a_n, b_n given by the formulas in Lecture 8. Here they are:

$$a_0 := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx \quad ,$$

and the numbers $a_1, a_2, a_3, \dots$ and $b_1, b_2, b_3, \dots$ are given by:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

In **real** life (as opposed to the artificial problems that you get as homework), $f(x)$ is usually only given *numerically* by some equipment interfaced with a computer (like every piece of equipment is nowadays, they are all "smart"). Then the computer computes the numbers $a_0, a_1, a_2, \dots$,

$b_1, b_2, b_3, \ldots$ using *numerical integration*, getting some concrete numbers. Of course, it can't do it for infinitely many of them, so it does it for the first one thousand coefficients, and this is good enough for all practical purposes.

If the function $f(x)$ is given *explicitly* by some formula, for example $f(x) = e^x$ then using Calc2 stuff it is sometimes possible to find "all" the coefficients a_n, b_n by deriving an *explicit* expression in n, but this is the exception to the rule. For many functions, for example $f(x) = e^{x^3}$, not even Maple can do it, and once again we do it numerically and only compute the first one thousand coefficients or whatever.

Finally, some functions are given via an *integral representation.* For example, the function

$$f(x) = \int_{-\infty}^{\infty} \cos(xy^2) e^{-y^4} \, dy \quad .$$

There is **no way** to find an "explicit" expression for $f(x)$ as an expression of x, but for practical purposes it is useful. For any input x, for example, $x = 2$, we plug-in $x = 2$ into the integral getting

$$f(2) = \int_{-\infty}^{\infty} \cos(2y^2) e^{-y^4} \, dy \quad ,$$

that *no one* knows how to do "exactly" but using numerical techniques for doing integrals, you can get a numerical answers. By doing it for $x = 0, 0.01, 0.02, 0.03 \ldots, 10000$, the computer can plot $f(x)$ very accurately.

Note that Integral Representations are very confusing. In the integral

$$\int_{-\infty}^{\infty} \cos(xy^2) e^{-y^4} \, dy \quad ,$$

two letters show up, x and y. The boss is y. It is the integration variable, because we have dy. The letter x is conceptually a mere number, that's why it is called a *parameter.*

The most famous Integral Representation is the Fourier-Integral Representation.

The **Fourier Integral Representation** of a function $f(x)$ defined on the real line $(-\infty, \infty)$ is given by

$$f(x) = \frac{1}{\pi} \int_0^{\infty} [\, A(\alpha) \cos \alpha x + B(\alpha) \sin \alpha x \,] \, d\alpha \quad ,$$

here α is the variable of integration and x is the *parameter.* The functions $A(\alpha)$ and $B(\alpha)$ are also given by integrals: where

$$A(\alpha) = \int_{-\infty}^{\infty} f(x) \cos \alpha x \, dx \quad ,$$

$$B(\alpha) = \int_{-\infty}^{\infty} f(x) \sin \alpha x \, dx \quad .$$

Now x is the variable of integration and α is the parameter.

Avoiding Gibberish

Here is an examples of gibberish. For a solution of a wave-equation problem, one student had

$$u(x,t) = \cos x \sin 3t \sin nt \quad .$$

n has no place here! n can only show up in solutions that have $\sum_{n=1}^{\infty}$ in front of them, and then it means that we have to add-up the terms for $n = 1$, $n = 2$, $n = 3$ for ever after. But **please** don't have a disembodied $\sin nt$. If you do, you will get **zero** points, since gibberish answers are worse than no answer, and even worse than the wrong answer.