Dr. Z.'s Calc5 Lecture 19 Handout:
Applications of the Laplace Transfrom for solving Partial Differential Equations

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Important Formula

Recall that the **Laplace Transform** of a function $f(t)$ of a **single** variable t ($t > 0$, t is usually time), is

$$\mathcal{L}\{f(t)\} \,=\, \int_0^\infty e^{-st} f(t)\, dt \,=\, F(s) \quad .$$

If we have a function of **two** variables $u(x,t)$, the Laplace Transform with respect to t is obviously

$$\mathcal{L}\{u(x,t)\} \,=\, \int_0^\infty e^{-st} u(x,t)\, dt \,=\, U(x,s) \quad .$$

In other words, we think of x as a *parameter* and consider $u(x,t)$ as a function of t alone.

Important Formula:

If $\mathcal{L}\{u(x,t)\} = U(x,s)$, then (by the formuals from Lecture 2, p.3)

$$\mathcal{L}\{\frac{\partial u}{\partial t}\} \,=\, sU(x,s) - u(x,0) \quad ,$$

$$\mathcal{L}\{\frac{\partial^2 u}{\partial t^2}\} \,=\, s^2U(x,s) - su(x,0) - u_t(x,0) \quad .$$

Of course

$$\mathcal{L}\{\frac{\partial u}{\partial x}\} \,=\, \frac{\partial U(x,s)}{\partial x}$$

$$\mathcal{L}\{\frac{\partial^2 u}{\partial x^2}\} \,=\, \frac{\partial^2 U(x,s)}{\partial x^2}$$

Problem 19.1: Find the Laplace Transform of the pde $9u_{xx} = u_{tt}$, $t > 0$.

Solution: Applying $\mathcal{L}$, we get:

$$\mathcal{L}\{9u_{xx}\} = \mathcal{L}\{u_{tt}\}$$

So

$$9U_{xx} = s^2U - su(x,0) - u_t(x,0) \quad .$$

Moving all the U stuff to the left we get, viewing $U(x,s)$ as a function of x, and considering (as we should) s as a parameter

$$9U''(x) - s^2U(x) = -su(x,0) - u_t(x,0) \quad .$$

Ans. to 19.1: $9U''(x) - s^2U(x) = -su(x,0) - u_t(x,0)$.

Problem 19.2: Solve the pde

$$4u_{xx} = u_{tt} \quad , \quad 0 < x < 2 \quad , \quad t > 0 \quad ,$$

subject to the **boundary-conditions**

$$u_x(0,t) = 0 \quad , \quad u_x(2,t) = 0 \quad , \quad t > 0 \quad ,$$

and the **initial conditions**

$$u(x,0) = 0 \quad , \quad u_t(x,0) = -\cos(\pi x/2) \quad , \quad 0 < x < 1 \quad .$$

Solution: First apply $\mathcal{L}$ to the pde getting

$$\mathcal{L}\{4u_{xx}\} = \mathcal{L}\{u_{tt}\} \quad , \quad 0 < x < 2 \quad .$$

Defining $U(x,s) = \mathcal{L}\{u(x,t)\}$, we have

$$4U''(x,s) = s^2U(x,s) - su(x,0) - u_t(x,0) \quad , \quad 0 < x < 2 \quad .$$

Or, abbreviating $U(x,s)$ to just $U(x)$, (s is a silent parameter)

$$4U''(x) - s^2U(x) = -su(x,0) - u_t(x,0) \quad .$$

Taking advantage of the initial conditions, $u(x,0) = 0$, $u_t(x,0) = -\cos(\pi x/2)$, this becomes:

$$4U'' - s^2U = \cos(\pi x/2) \quad .$$

Because of the boundary conditions, $U'(0) = 0, U'(2) = 0$.

We have to solve this ode. Regarding the homogeneous part, the general solution of $4U'' - s^2U = 0$ which is the same as $U'' - (s^2/4)U = 0$ is

$$c_1e^{(s/2)x} + c_2e^{-(s/2)x} \quad .$$

Regarding a **particular solution** we write

$$U(x) = C\cos(\pi x/2) \quad .$$

So

$$U''(x) = -C(\pi/2)^2\cos(\pi x/2) \quad .$$

Putting this into the ode, we get

$$-4C(\pi/2)^2\cos(\pi x/2) - s^2C\cos(\pi x/2) = \cos(\pi x/2)$$

Factoring out the C

$$C[-\pi^2 - s^2]\cos(\pi x/2) = \cos(\pi x/2)$$

that implies

$$C[-\pi^2 - s^2] = 1 \quad .$$

Solving for C we get

$$C = -\frac{1}{\pi^2 + s^2} \quad .$$

So the general solution of the ode is

$$U(x) = -\frac{1}{\pi^2 + s^2}\cos(\pi x/2) + c_1 e^{(s/2)x} + c_2 e^{-(s/2)x} \quad 0 < x < 2 \quad .$$

We still need to find the numbers c_1 and c_2, using the boundary conditions $U'(0) = 0, U'(2) = 0$.

$$U'(x) = \frac{1}{\pi^2 + s^2}(\pi/2)\sin(\pi x/2) + c_1(s/2)e^{(s/2)x} + c_2(-s/2)e^{-(s/2)x}$$

So

$$U'(0) = \frac{1}{\pi^2 + s^2}(\pi/2)\sin(0) + c_1(s/2)e^0 + c_2(-s/2)e^0 = s/2(c_1 - c_2)$$

$$U'(2) = \frac{1}{\pi^2 + s^2}(\pi/2)\sin(\pi) + c_1(s/2)e^s + c_2(-s/2)e^{-s} = c_1(s/2)e^s + c_2(-s/2)e^{-s} \quad .$$

So we have to solve the linear system of equations with two equations and two unknowns.

$$c_1 - c_2 = 0 \quad , \quad c_1(s/2)e^s + c_2(-s/2)e^{-s} = 0 \quad ,$$

whose solution is $c_1 = 0, c_2 = 0$. Going back to the general solution, and substituting $c_1 = 0, c_2 = 0$, we get

$$U(x,s) = \frac{-1}{\pi^2 + s^2}\cos(\pi x/2) \quad . \quad 0 < x < 2 \quad .$$

To recover $u(x,t)$ we take the **inverse Laplace Transfom**

$$u(x,t) = \mathcal{L}^{-1}\{\frac{-1}{\pi^2 + s^2}\cos(\pi x/2)\} = -\cos(\pi x/2)\mathcal{L}^{-1}\{\frac{1}{\pi^2 + s^2}\} = -\cos(\pi x/2)\frac{1}{\pi}\sin(\pi t) = -\frac{1}{\pi}\cos(\pi x/2)\sin(\pi t) \quad .$$

Ans. to 19.2: $u(x,t) = -\frac{1}{\pi}\cos(\pi x/2)\sin(\pi t)$.

Problem 19.3: Solve the pde

$$u_{xx} - 10 = u_{tt} \quad , x > 0 \quad , \quad t > 0 \quad ,$$

subject to the **boundary-conditions**

$$u(0,t) = 0 \quad , \quad u_x(\infty, t) = 0 \quad , \quad t > 0 \quad ,$$

and the **initial conditions**

$$u(x,0) = 0 \quad , \quad u_t(x,0) = 0 \quad , \quad x > 0 \quad .$$

Solution: Applying $\mathcal{L}$, and calling $\mathcal{L}\{u(x,t)\} = U(x,s)$, we have

$$\mathcal{L}\{u_{xx} - 10\} = \mathcal{L}\{u_{tt}\} \quad , x > 0 \quad .$$

$$U''(x) - \frac{10}{s} = s^2 U(x) - s u(x,0) - u_t(x,0) \quad .$$

Because $u(x,0) = 0$ and $u_t(x,0) = 0$, we have

$$U''(x) - \frac{10}{s} = s^2 U(x) \quad .$$

So we have to solve the ode, in x,

$$U'' - s^2 U = \frac{10}{s}$$

Trying how the **particular solution** $U(x) = C$ we get

$$0 - s^2 C = \frac{10}{s} \quad ,$$

so

$$C = -\frac{10}{s^3} \quad .$$

The general solution of the **homogeneous** version: $U'' - s^2 U =$ is $c_1 e^{xs} + c_2 e^{-xs}$ So the **general solution** of the ode is:

$$U(x) = -\frac{10}{s^3} + c_1 e^{xs} + c_2 e^{-xs} \quad .$$

Now it is time to take care of the **boundary conditions** that $U'(\infty) = 0$ and $U(0) = 0$. Differentiating $U(x)$ we get

$$U'(x) = c_1 s e^{xs} - c_2 s e^{-xs} \quad .$$

The only way that $U'(\infty) = 0$ is if $c_1 = 0$, so

$$U(x) = -\frac{10}{s^3} + c_2 e^{-xs} \quad .$$

Using $x = 0$ yields

$$U(0) = -\frac{10}{s^3} + c_2 e^{-0 \cdots} = -\frac{10}{s^3} + c_2 \, = 0 \, .$$

So

$$c_2 = \frac{10}{s^3} \quad .$$

Going back to $U(x)$ we have

$$U(x) = -\frac{10}{s^3} + \frac{10}{s^3} e^{-xs} \quad .$$

Now it is time to **go back** and apply $\mathcal{L}^{-1}$:

$$u(x,t) = \mathcal{L}^{-1}\{U(x,s)\} = \mathcal{L}^{-1}\{-\frac{10}{s^3}\} + \mathcal{L}^{-1}\{\frac{10}{s^3} e^{-xs}\} = -5t^2 + 5(t-x)^2 \mathcal{U}(t-x) \quad .$$

(by the **Second Translation Theorem**).

Ans. to 19.4: $-5t^2 + 5(t-x)^2 \mathcal{U}(t-x)$.