Dr. Z.'s Calc5 Lecture 18 Handout: Laplace's Equation in Polar Coordinates

By Doron Zeilberger

Version of Nov. 14, 2011 (thanks to Jon Pita, who corrected a minor typo).

Important Formula

The Laplacian Equation in two dimensions

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u(x,y) = 0 \quad ,$$

phrased in the usual **rectangular coordinates** (x, y). becomes, in **polar coordinates** (r, θ)

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right) u(r,\theta) = 0 \quad .$$

Reminder: Polar → Rectangular: $x = r\cos\theta, y = r\sin\theta$; Rectangular → Polar: $r = \sqrt{x^2+y^2}, \theta = \tan^{-1}\frac{y}{x}$.

Important ODE:

The **Cauchy-Euler** differential equation

$$r^2 R''(r) + rR'(r) - n^2 R(r) = 0 \quad ,$$

has the **general solution**

$$R(r) = C_1 r^n + C_2 r^{-n} \quad ,$$

when $n > 0$. When $n = 0$, the general solution is $R(r) = C_1 + C_2 \ln r$.

Problem 18.1: Find the general expression, in polar coordinates, for the steady-state temperature $u(r, \theta)$ in a circular plate of radius 5, if the temperature on the circumference $r = 5$ is given by $u(5, \theta) = f(\theta)$ for some **periodic** function $f(\theta)$ of period 2π (in other words, $f(\theta + 2\pi) = f(\theta)$ for every θ).

Solution: The pde satisfied by $u(r, \theta)$ is **Laplce's Equation**, and since we are working in polar coordinates, the pde is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad .$$

We are first looking for **separable solutions**, i,e. those lucky functions of **two** variables, $u(r, \theta)$ that happen to be a **product** of a function of r alone, let's call it $R(r)$, and a function of θ alone, let's call it $\Theta(\theta)$. (Note, most functions of (r, θ) are not so lucky, for example $r + \theta$). So

$$u(r, \theta) = R(r)\Theta(\theta) \quad .$$

Plugging this into the pde, noting that

$$u_{rr} = R''(r)\Theta(\theta) \quad , \quad u_r = R'(r)\Theta(\theta) \quad , \quad u_{\theta\theta} = R(r)\Theta''(\theta) \quad ,$$

we get

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0 \quad .$$

Divide this by $R(r)\Theta(\theta)$, we get

$$\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} = 0 \quad .$$

Multiplying by r^2:

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0 \quad .$$

Leaving the r stuff on the left, and moving the θ stuff to the right, we get:

$$\frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} \quad .$$

The left side is **independent** of the variable θ, while the right side is independent of the variable r. Since the left side **equals** the right side, **both sides** are independent of **both variables**, in other words, both sides are equal to some **constant** number, let's call it λ, so we have

$$\frac{r^2R''(r) + rR'(r)}{R(r)} = \lambda$$

$$-\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda \quad .$$

Simplifying, we get the two odes

$$r^2R''(r) + rR'(r) - \lambda R(r) = 0 \quad .$$

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0 \quad .$$

Let's first tackle the second ode. If λ is negative we get combinations of exponential functions (or sine-hyperbolic and cosine-hyperbolic functions) and there is no way that they can be periodic. If $\lambda = 0$ then the general solution is $\Theta(\theta) = c_1 + c_2\theta$. Of course if $c_2 \neq 0$ there is no way that it can be periodic (it is a straight line!), so in that case we have $c_2 = 0$ and

$$\Theta(\theta) = c_1 \quad ,$$

and this is of course periodic (since any constant is!). If λ is positive, as can write, for convenience, $\lambda = \alpha^2$ and we get

$$\Theta''(\theta) + \alpha^2\Theta(\theta) = 0 \quad ,$$

whose general solution is

$$\Theta(\theta) = c_1\cos(\alpha\theta) + c_2\sin(\alpha\theta) \quad .$$

Now what's nice about the trig. functions $\sin\theta$ and $\cos\theta$ is that they are **periodic** of period 2π. Of course the same is true for $\sin(n\theta)$ and $\cos(n\theta)$ for *any* **integer** n, but if if α is **not** an integer, then $\sin(\alpha\theta)$ and $\cos(\alpha\theta)$ are not periodic of period 2π (often they are periodic but with a different period, we need the period to be 2π). So this means that α must be a pos. integer, and we get the options

$$\Theta(\theta) = c_1 \quad ; \quad \Theta(\theta) = c_1\cos(n\theta) + c_2\sin(n\theta) \quad (n = 1, 2, 3, \dots) \quad .$$

We now need to find the $R(r)$ counterpart to $\lambda = -n^2$. The ode is the **Cauchy-Euler** ode

$$r^2 R''(r) + rR'(r) + n^2 R(r) = 0 \quad .$$

whose general solution is

$$R(r) = c_3 + c_4 \ln r \quad , (n = 0) \quad ; \quad R(r) = c_3 r^n + c_4 r^{-n} \quad , (n = 1, 2, 3, \dots) \quad .$$

But $\ln r$ and $r^{-n} = 1/r^n$ both **blow up** at $r = 0$, so must be abandoned. So $c_4 = 0$ in either case and

$$R(r) = c_3 \quad (n = 0) \quad ; \quad R(r) = c_3 r^n \quad , (n = 1, 2, 3, \dots) \quad .$$

Combining, the **separable** solutions are

$$u(r, \theta) = c_1 c_3 \quad (n = 0) \quad ; \quad (c_1\cos(n\theta) + c_2\sin(n\theta))c_3 r^n \quad (n = 1, 2, 3, \dots) \quad .$$

Renaming constants $c_1 c_3$ and $c_2 c_3$ we get

$$u(r, \theta) = A_0 \quad (n = 0) \quad ; \quad (A_n\cos(n\theta) + B_n\sin(n\theta))r^n \quad (n = 1, 2, 3, \dots) \quad .$$

Now it is time to look at the **boundary condition** $u(5, \theta) = f(\theta)$. If you are lucky, and instead of the **general** (symbolic) $f(\theta)$ you are given a **specific** function, and not just *any* specific function (like, for example θ^2), but one that looks like either C, or $C\cos(n_0\theta)$ or $C\sin(n_0\theta)$ for some specific integer n_0, for some specific constant C. The the answer would simply be $u(r, \theta) = C$, or $u(r, \theta) = C\cos(n_0\theta)/5^n$ or $u(r, \theta) = C\sin(n_0\theta)/5^n$, respectively. But otherwise we must use the **principle of superposition**. The following is a solution of the pde so far (not yet taking care of $u(5, \theta) = f(\theta)$)

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} (A_n\cos(n\theta) + B_n\sin(n\theta))r^n \quad ,$$

for *any* choice of the infinitely many numbers $A_0, A_1, A_2 \dots ; B_1, B_2, \dots$. Now it is time to use the additional **data** that $u(5, \theta) = f(\theta)$. Plugging-in $r = 5$, we get

$$u(5, \theta) = A_0 + \sum_{n=1}^{\infty} (A_n\cos(n\theta) + B_n\sin(n\theta))5^n \quad ,$$

So

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} 5^n A_n\cos(n\theta) + 5^n B_n\sin(n\theta) \quad .$$

Recall that the **full Fourier series** of a periodic function of period 2π is:

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta \quad ,$$

where

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(\theta)\, d\theta \quad ,$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta\, d\theta \quad ,$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta\, d\theta \quad ,$$

So

$$A_0 = \frac{a_0}{2} = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta \quad , \quad 5^n A_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta\, d\theta \quad , \quad 5^n B_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta\, d\theta \quad .$$

yielding

$$A_0 = \frac{2}{\pi} \int_0^{2\pi} f(\theta) \quad , \quad A_n = \frac{1}{5^n \pi} \int_0^{2\pi} f(\theta) \cos n\theta\, d\theta \quad , \quad B_n = \frac{1}{5^n \pi} \int_0^{2\pi} f(\theta) \sin n\theta\, d\theta \quad .$$

Putting it together, the **answer** is:

$$u(r, \theta) = A_0 \quad + \sum_{n=1}^{\infty} (A_n \cos(n\theta) + B_n \sin(n\theta)) r^n \quad ,$$

where the numbers $A_0, A_1, A_2, \ldots, B_1, B_2, \ldots$ are given by the above formulas. This is the **answer**.

Variation on a Theme. Some problems are about the steady-state temperature in a **semi-circular plate**. where the bottom is held at temperature 0, and the top (circular arc) is given by $u(c, \theta) = f(\theta)$ for some function on $[0, \pi]$. Then the solution is very similar to the above, but a little simpler. The Θ equation is

$$\Theta''(\theta) + \lambda \Theta(\theta) = 0 \quad , \Theta(0) = 0 \quad , \quad \Theta(\pi) = 0 \quad ,$$

so the $\cos n\theta$ are out of the picture and you have a **Fourier-sine** series on $[0, \pi]$.