Dr. Z.'s Calc5 Lecture 17 Handout: Laplace's Equation

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Important Problem (Laplace's Equation in a Rectangle)

Solve

$$u_{xx} + u_{yy} = 0 \quad , \quad 0 < x < a \quad , \quad 0 < y < b \quad ,$$

subject to various types of boundary conditions, involving the function itself or its derivatives on the four side.

Problem 17.1 Solve

$$u_{xx} + u_{yy} = 0 \quad , \quad 0 < x < \pi \quad , \quad 0 < y < 1 \quad ,$$

subject to

$$u_x(0, y) = 0 \quad , \quad u_x(\pi, y) = 0 \quad , \quad 0 < y < 1 \quad ;$$

$$u(x, 0) = 0 \quad , \quad u(x, 1) = f(x) \quad , \quad 0 < x < \pi \quad .$$

Solution: We first look for **separable solutions** of the type

$$u(x, y) = X(x)Y(y) \quad .$$

Since

$$u_{xx} = X''(x)Y(y) \quad , \quad u_{yy} = X(x)Y''(y) \quad ,$$

we have

$$X''(x)Y(y) + X(x)Y''(y) = 0 \quad .$$

Dividing by $X(x)Y(y)$, we have

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0 \quad ,$$

so

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} \quad .$$

The left side does not depend on y, and the right side does not depend on x, since they are the same, **neither** of them depends on x or y, so they are **both** equal to the **same** constant, let's call it $-\lambda$. We have

$$\frac{X''(x)}{X(x)} = -\lambda \quad ,$$

$$\frac{Y''(y)}{Y(y)} = -\lambda \quad .$$

Leading to two **odes**:

$$X''(x) + \lambda X(x) = 0 \quad ,$$

$$Y''(y) - \lambda Y(y) = 0 \quad .$$

Now it is time to look at the homogeneous boundary conditions (those whose right hand side is 0). Since $u(x,y) = X(x)Y(y)$, $u_x(x,y) = X'(x)Y(y)$, and

$$u_x(0,y) = 0 \quad , \quad 0 < y < 1 \quad ,$$

means

$$X'(0)Y(y) = 0 \quad .$$

Since the function $Y(y)$ better not be zero (or else we get the trivial, zero, solution), we must have:

$$X'(0) = 0 \quad .$$

Another boundary condition is:

$$u_x(\pi,y) = 0 \quad , \quad 0 < y < 1 \quad ,$$

means

$$X'(\pi)Y(y) = 0 \quad ,$$

so

$$X'(\pi) = 0 \quad .$$

For future reference, $u(x,0) = 0$ means

$$X(x)Y(0) = 0 \quad ,$$

so $Y(0) = 0$.

We first have to solve the **Sturm-Liouville** system

$$X''(x) + \lambda X(x) = 0 \quad , \quad 0 < x < \pi \quad , \quad X'(0) = 0 \quad , \quad X'(\pi) = 0 \quad .$$

Case I: $\lambda < 0$. Writing $\lambda = -\alpha^2$, we get

$$X''(x) - \alpha^2 X(x) = 0 \quad , \quad 0 < x < \pi \quad , \quad X'(0) = 0 \quad , \quad X'(\pi) = 0 \quad .$$

So

$$X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x} \quad ,$$

entailing that

$$X'(x) = c_1 \alpha e^{\alpha x} - c_2 \alpha e^{-\alpha x} \quad ,$$

that in turn, lead to:

$$X'(0) = c_1\alpha - c_2\alpha \quad , \quad X'(\pi) = c_1 \alpha e^{\alpha\pi} - c_2 \alpha e^{-\alpha\pi} \quad .$$

We have to find real numbers c_1, c_2 such that

$$c_1\alpha - c_2\alpha = 0 \quad , \quad c_1 \alpha e^{\alpha\pi} - c_2 \alpha e^{-\alpha\pi} = 0 \quad .$$

From the first equation $c_1 = c_2$ (since $\alpha \neq 0$), so $c_1 \alpha e^{\alpha\pi} - c_1 \alpha e^{-\alpha\pi} = 0$ so $c_1(\alpha e^{\alpha\pi} - \alpha e^{-\alpha\pi}) = 0$ and we get $c_1 = 0$, and hence also $c_2 = 0$, so we only got the **trivial solution**, that does not count.

Case II: $\lambda = 0$.

$$X''(x) = 0 \quad , \quad 0 < x < \pi \quad , \quad X'(0) = 0 \quad , \quad X'(\pi) = 0 \quad .$$

So $X(x) = c_1 + c_2 x$, $X'(x) = c_2$, $X'(0) = c_2$, $X'(\pi) = c_2$, so $c_2 = 0$, and we got that $X(x) = c_1$ is a solution. The counterpart ode for $Y(y)$ is $Y''(y) = 0$ whose general solution is $Y(y) = c_3 + c_4 y$, so $u(x, y) = X(x)Y(y) = c_1(c_3 + c_4 y)$. Using $u(x, 0) = 0$ gives $c_3 = 0$ so $u(x, y) = c_1 c_3 y$. Renaming $c_1 c_3$, A_0, we get only one solution from Case II $u(x, y) = A_0 y$.

Case III: $\lambda > 0$. Writing $\lambda = \alpha^2$, we get

$$X''(x) + \alpha^2 X(x) = 0 \quad , \quad 0 < x < \pi \quad , \quad X'(0) = 0 \quad , \quad X'(\pi) = 0 \quad .$$

So $X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$, $X'(x) = -\alpha c_1 \sin(\alpha x) + \alpha c_2 \cos(\alpha x)$, $X'(0) = \alpha c_2$, $X'(\pi) = -\alpha c_1 \sin(\alpha\pi) + \alpha c_2 \cos(\alpha\pi)$. So $c_2 = 0$ (since $\alpha \neq 0$), and $\sin(\alpha\pi) = 0$.

We have to solve the equation in α,

$$\sin(\alpha\pi) = 0 \quad .$$

There are infinitely many solutions $\alpha = 1, 2, \ldots$, in general $\alpha = n$ for any positive integer, (the case $\alpha = 0$ we already have from above and besides in case II we assume $\alpha > 0$). The corresponding solution is

$$X(x) = c_1 \cos(nx) \quad , \quad n = 1, 2, \ldots$$

We now need to find the counterpart $Y(y)$ for each of $\lambda = n^2$ (for $n = 1, 2, 3, \ldots$).

$$Y''(y) - n^2 Y(y) = 0 \quad .$$

The general solution is

$$Y(y) = c_3 \sinh ny + c_4 \cosh ny \quad .$$

Remember $u(x, y) = X(x)Y(y)$, so

$$u(x, y) = c_1 \cos(nx)(c_3 \sinh(ny) + c_4 \cosh(ny)) \quad .$$

Using the third boundary conditions $u(x, 0) = 0$, we get

$$0 = u(x, 0) = c_1 \cos(nx)(c_3 \sinh 0 + c_4 \cosh 0) = c_1 c_4 \cos(nx) \quad ,$$

so $c_4 = 0$ and

$$u(x, y) = c_1 c_3 \cos(nx) \sinh(ny) \quad .$$

We rename $c_1 c_3$ to A_n and have the solution

$$u(x,y) = A_n \cos(nx) \sinh(ny) \quad ,$$

for an **arbitrary** constant A_n.

Now it is time to take care of the last boundary condition, $u(x,1) = f(x)$. If we are lucky, and $f(x)$ happens to be **exactly** of the form

$$f(x) = C \cos(nx) \quad ,$$

for some specific integer n, and some specific constand C, then

$$u(x,1) = A_n \cos(nx) \sinh(n) = C \cos(nx) \quad .$$

and we get $A_n = C/sinh(n)$, and the final answer would have been

$$u(x,y) = \frac{C}{\sinh(n)} \cos(nx) \sinh(ny) \quad .$$

Alas, for a general $f(x)$ that is **not** a constant multiple of a pure cosine function, we must go on and use the **principle of superposition** and **Fourier series**.

By the principle of superposition, the following function

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos(nx) \sinh(ny) \quad ,$$

is a solution to the pde plus the first three boundary conditions, for *every* choice of constants $A_0, A_1, A_2, A_2 \ldots$, so without the last boundary condition, there are $\infty^\infty = \infty$ answers. Now it is time to impose the fourth boundary condtion

$$u(x,1) = f(x) \quad .$$

So

$$f(x) = u(x,1) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) \sinh(n) = A_0 + \sum_{n=1}^{\infty} (A_n \sinh(n)) \cos(nx) \quad .$$

So we need the **Fourier cosine series** of Lecture 9.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad ,$$

where

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx \quad ,$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx \quad ,$$

So, comparing to the situations that we have now

$$2A_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx \quad ,$$

$$A_n \sinh(n) = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx \quad ,$$

Doing the algebra, gives

$$A_0 = \frac{1}{\pi}\int_0^\pi f(x)\,dx$$

$$A_n = \frac{2}{\pi\sinh(n)}\int_0^\pi f(x)\cos nx\,dx \quad .$$

So the **final answer** is that the solution $u(x,y)$ of our boundary-value problem is

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos(nx)\sinh(ny) \quad ,$$

where

$$A_0 = \frac{1}{\pi}\int_0^\pi f(x)\,dx \quad , \quad A_n = \frac{2}{\pi\sinh(n)}\int_0^\pi f(x)\cos nx\,dx \quad .$$

This is the **answer**. Another way of writing the answer is the following hairy formula

Ans. to Problem 17.1:

$$u(x,y) = \left(\frac{1}{\pi}\int_0^\pi f(x)\,dx\right) y + \sum_{n=1}^{\infty}\left(\frac{2}{\pi\sinh(n)}\int_0^\pi f(x)\cos nx\,dx\right)\cos(nx)\sinh(ny) \quad .$$

Important Property: Boundary Superposition Principle

If you have a complicated so-called *Dirichlet* boundary value problem

$$u_{xx} + u_{yy} = 0 \quad , \quad 0 < x < a \quad , \quad 0 < y < b$$

$$u(0,y) = F(y) \quad , \quad u(a,y) = G(y) \quad , \quad 0 < y < b \quad .$$

$$u(x,0) = f(x) \quad , \quad u(x,b) = g(x) \quad , \quad 0 < x < a \quad .$$

You **break-it up** into two problems:

First Problem: Find the solution, let's call it $u_1(x,y)$ satisfying

$$(u_1)_{xx} + (u_1)_{yy} = 0 \quad , \quad 0 < x < a \quad , \quad 0 < y < b$$

$$u_1(0,y) = 0 \quad , \quad u_1(a,y) = 0 \quad , \quad 0 < y < b \quad .$$

$$u_1(x,0) = f(x) \quad , \quad u_1(x,b) = g(x) \quad , \quad 0 < x < a \quad .$$

Second Problem: Find the solution, let's call it $u_2(x, y)$ satisfying

$$(u_2)_{xx} + (u_2)_{yy} = 0 \quad , \quad 0 < x < a \quad , \quad 0 < y < b$$

$$u_2(0, y) = F(y) \quad , \quad u_2(a, y) = G(y) \quad , \quad 0 < y < b \quad .$$

$$u_2(x, 0) = 0 \quad , \quad u_2(x, b) = 0 \quad , \quad 0 < x < a \quad .$$

Once you solved these (already complicated!) two problems, the **final** solution, to the original problem, is simply

$$u(x, y) = u_1(x, y) + u_2(x, y) \quad .$$

In other words, just add them up!