Dr. Z.'s Calc5 Lecture 15 Handout: The Heat Equation

By Doron Zeilberger

The **Heat Equation** is the following **pde**

$$k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad , \quad 0 < x < L \quad , \quad t > 0$$

for some **constant** k, that must be positive. In **subscript notation** it is:

$$k u_{xx} = u_t \quad .$$

There are lots of **boundary conditions**. The most common one is

$$u(0,t) = 0 \quad , \quad u(L,t) = 0 \quad , \quad t > 0$$

(both ends are held at temperature 0). Another one is:

$$u_x(0,t) = 0 \quad , \quad u_x(L,t) = 0 \quad , \quad t > 0$$

(both ends are insulated), and any combination. The **initial condition** is always the same

$$u(x,0) = f(x) \quad , \quad 0 < x < L \quad ,$$

for some function $f(x)$ describing the temperature along the rod at the very beginnig.

Important Formula

The solution of

$$k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad , \quad 0 < x < L \quad , \quad t > 0$$

subject to

$$u(0,t) = 0 \quad , \quad u(L,t) = 0 \quad , \quad t > 0$$

$$u(x,0) = f(x) \quad , \quad 0 < x < L \quad ,$$

is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k(n^2\pi^2/L^2)t} \sin \frac{n\pi}{L} x \quad ,$$

where

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx \quad .$$

Important Special Case: It is always posible to transfer a problem from $[0, L]$ to $[0, \pi]$. The above hairy formulas become simpler when $L = \pi$.

The solution of

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad , \quad 0 < x < \pi \quad , \quad t > 0 \quad ,$$

subject to

$$u(0,t) = 0 \quad , \quad u(\pi,t) = 0 \quad , t > 0$$

$$u(x,0) = f(x) \quad , \quad 0 < x < \pi \quad ,$$

is

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} A_n e^{-kn^2 t} \sin nx \quad ,$$

where

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx \quad .$$

How to go from the general case to the special case

It is always possible to go from the $[0, L]$ interval to the easier $[0, \pi]$ by the **change of variables**

$$v(x,t) = u((L/\pi)x, t) \quad ,$$

get a boundary value problem for $v(x,t)$, in $[0, \pi]$, solve it and go back with

$$u(x,t) = v((\pi/L)x, t) \quad .$$

Of course, $f(x)$ turns into $f(xL/\pi)$, and the constant k becomes $(L/\pi)^2 k$.

Problem 15.1: Derive from first principle the above formula for the interval $[0, \pi]$.

Solution: We first use **separation of variables**, and write

$$u(x,t) = X(x)T(t) \quad ,$$

and look for such solutions. Since

$$u_{xx} = X''(x)T(t) \quad , u_t = X(x)T'(t) \quad ,$$

we have

$$kX''(x)T(t) = X(x)T'(t) \quad , 0 < x < \pi, t > 0$$

Moving all the x stuff to the left and all the t stuff to the right we get

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)}$$

Since the left side does not depend on t and the right side does not depend on x, and they are equal to each other, neither of them depend on x or t, in other words they are both equal to (the same!) **constant**. It is easy to see that when that constant happens to be zero or positive there is no way the boundary conditions can be met with a non-trivial (i.e. non-zero) solution. So we can call that common constant (the so-called separation constant) $-\alpha^2$ (for convenience, it is good to have α^2 rather than α, also this way we are guaranteed that the separation constant is always negative, since $-\alpha^2$ is always negative). So

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = -\alpha^2 \quad .$$

Meaning that

$$\frac{X''(x)}{X(x)} = -\alpha^2 \quad , \quad \frac{T'(t)}{kT(t)} = -\alpha^2 \quad .$$

Leading to **two odes**.

$$X''(x) + \alpha^2 X(x) = 0 \quad , \quad T'(t) + k\alpha^2 T(t) = 0 \quad .$$

By Calc4 stuff, the **general solution** of the first one is

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$$

for some constants c_1, c_2, and for the second one is

$$T(t) = c_3 e^{-k\alpha^2 t} \quad .$$

So the following are all solutions of the Heat Equation (but we have not yet looked at the initial and bounary conditions), for *any* real number α, you name it!.

$$u(x,t) = (c_1 \cos \alpha x + c_2 \sin \alpha x) c_3 e^{-k\alpha^2 t} \quad .$$

Now it is time to look at the **boundary conditions**. When $x = 0$, we have

$$u(0,t) = (c_1 \cos \alpha 0 + c_2 \sin \alpha 0) c_3 e^{-k\alpha^2 t} = c_1 c_3 e^{-k\alpha^2 t} \quad .$$

But $u(0,t) = 0$ so we need $c_1 = 0$ (of course we don't want $c_3 = 0$). Now $u(x,t)$ becomes

$$u(x,t) = A(\sin \alpha x) e^{-k\alpha^2 t}$$

where we replaced $c_2 c_3$ by the constant name A. (This is always legal since we are talking **arbitrary constants**).

Now it is time to use the second boundary condition $u(\pi, t) = 0$:

$$u(\pi, t) = A \sin(\alpha \pi) e^{-k\alpha^2 t} = 0 \quad .$$

Since A should not be 0 (or else we get nothing!) and $e^{-k\alpha^2 t}$ is **never** 0, since forces us to have

$$\sin(\alpha \pi) = 0 \quad .$$

This means that $\alpha = 1$ or $\alpha = 2$ etc., in other words α **must** be a **positive integer** (or negative, but since $\sin(-nx) = -\sin(nx)$ and $(-n)^2 = n^2$ they don't give any thing new).

So we found **infinitely** many solutions, let's call them $u_n(x,t)$

$$u_n(x,t) = A_n(\sin nx)e^{-kn^2t} \quad .$$

So far we have not yet imposed the **initial condition**

$$u(x,0) = f(x) \quad , \quad 0 < x < \pi \quad .$$

When we plug-in $t = 0$ in the formula for $u_n(x,t)$ we get

$$u_n(x,0) = A_n(\sin nx) \quad .$$

If you are **lucky** and $f(x)$ happens to be of that **form** i.e. $f(x)$ is some constant times $\sin x$, or some constant times $\sin 2x$, or in general some constant times $\sin nx$ for some **integer** n, then you are done, and $u_n(x,t)$ for the appropriate A_n is the **answer**. For example, if $f(x)$ happens to be

$$f(x) = 113 \sin 11x \quad ,$$

then you are **done!**. The final answer, in that case, is simply

$$u(x,t) = 113(\sin 11x)e^{-k11^2t} = 113(\sin 11x)e^{-121kt} \quad .$$

End of story! But if $f(x)$ is something else, for example $f(x) = x(\pi - x)$ or $f(x) = x(\pi - x)e^x$ then you are out of luck, and must go on.

We found *infinitey* many solutions for the pde+boundary values, namely $u_n(x,t) = A_n \sin nx e^{-kn^2t}$ for $n = 1, 2, 3, \ldots$. By the **principle of superposition**, the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n(\sin nx)e^{-kn^2t} \quad ,$$

for **any** choice of constants $A_0, A_1, A_2, A_3, \ldots$. Plugging-in $t = 0$ we get

$$u(x,0) = \sum_{n=1}^{\infty} A_n(\sin nx)e^{-kn^2 \cdot 0} = \sum_{n=1}^{\infty} A_n \sin nx \quad .$$

Since $u(x,0) = f(x)$, we need to find constants $A_0, A_1, A_2, A_3, \ldots$.

$$f(x) = \sum_{n=1}^{\infty} A_n \sin nx \quad .$$

But this is the **Fourier sine series** from Lecture 9. So we can recover the constants A_n by the formula

$$A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \quad .$$

Problem 15.2: Using the **ready-made formula** (don't do it from scratch) solve the boundary value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad , \quad 0 < x < \pi \quad , \quad t > 0 \quad ,$$

$$u(0,t) = 0 \quad , \quad u(\pi,t) = 0 \quad , t > 0$$

$$u(x,0) = x(\pi - x) \quad , \quad 0 < x < \pi \quad ,$$

Solution of 15.2: Here $k = 1$,$L = \pi$ (so that we can use the simpler formula), and $f(x) = x(\pi - x)$. All we need is find

$$A_n = \frac{2}{\pi} \int_0^{\pi} x(\pi - x) \sin nx$$

From Maple, or from integration tables,

$$A_n = \frac{2}{\pi} \int_0^{\pi} x(\pi - x) \sin nx = \frac{4(1 - (-1)^n)}{n^3} \quad .$$

So we have

$$u(x,t) = \sum_{n=1}^{\infty} (\sin nx) \frac{4(1 - (-1)^n)}{n^3} e^{-n^2 t} \quad ,$$

Since $1 - (-1)^n = 0$ when n is even and $1 - (-1)^n = 2$ if n is odd, this can be simplified to:

$$u(x,t) = 8 \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} e^{-(2k+1)^2 t} \sin(2k+1)x \quad .$$

Problem 15.3: Derive from first principle the solution of the following type of boundary value problem

$$k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad , \quad 0 < x < \pi \quad , \quad t > 0 \quad ,$$

subject to

$$u_x(0,t) = 0 \quad , \quad u_x(\pi,t) = 0 \quad , t > 0$$

$$u(x,0) = f(x) \quad , \quad 0 < x < \pi \quad ,$$

Solution: The beginning is the same. We first use **separation of variables**, and write

$$u(x,t) = X(x)T(t) \quad ,$$

and look for such solutions. Since

$$u_{xx} = X''(x)T(t) \quad , u_t = X(x)T'(t) \quad ,$$

we have

$$kX''(x)T(t) = X(x)T'(t) \quad , 0 < x < \pi, t > 0$$

Moving all the x stuff to the left and all the t stuff to the right we get

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)}$$

Since the left sides does not depend on t and the right side does not depend on x, and they are equal to each other, neither of them depend on x or t, in other words they are both equal to (the same!) **constant**. It is easy to see that when that constant happens to be zero or positive there is no way the boundary conditions can be met with a non-trivial (i.e. non-zero) solution. So we can call that common constant (the so-called separation constant) $-\alpha^2$ (for convenience, it is good to have α^2 rather than α, also this way we are guaranteed that the separation constant is always negative, since $-\alpha^2$ is always negative. So

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = -\alpha^2 \quad .$$

Meaning that

$$\frac{X''(x)}{X(x)} = -\alpha^2 \quad , \quad \frac{T'(t)}{kT(t)} = -\alpha^2 \quad .$$

Leading to **two odes**.

$$X''(x) + \alpha^2 X(x) = 0 \quad , \quad T'(t) + k\alpha^2 T(t) = 0 \quad .$$

By Calc4 stuff, the **general solution** of the first one is

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$$

for some constants c_1, c_2, and for the second one is

$$T(t) = c_3 e^{-k\alpha^2 t} \quad .$$

So the following are all solutions of the Heat Equation (but we have not yet looked at the initial and bounary conditions), for *any* real number α, you name it!.

$$u(x,t) = (c_1 \cos \alpha x + c_2 \sin \alpha x) c_3 e^{-k\alpha^2 t} \quad .$$

Since **now** the boundary conditions involve u_x rather than u, we need to differentiate with respect to x:

$$u_x(x,t) = (-c_1 \alpha \sin \alpha x + c_2 \alpha \cos \alpha x) c_3 e^{-k\alpha^2 t} \quad .$$

Now it is time to look at the **boundary conditions**. When $x = 0$, we have

$$u_x(0,t) = (-c_1 \alpha \sin \alpha 0 + c_2 \alpha \cos \alpha 0) c_3 e^{-k\alpha^2 t} = c_2 c_3 \alpha e^{-k\alpha^2 t}$$

But $u_x(0,t) = 0$ so we need $c_2 = 0$ (of course we don't want $c_3 = 0$). Now $u(x,t)$ becomes

$$u(x,t) = A(\cos \alpha x) e^{-k\alpha^2 t}$$

where we renamed replace $c_1 c_3$ by the constant name A. (This is always legal since we are talking **arbitrary constants**). Now it is time to use the second boundary condition $u_x(\pi, t) = 0$:

$$u_x(x,t) = A(-\alpha)(\sin \alpha x)e^{-k\alpha^2 t}$$

$$u_x(\pi,t) = A(-\alpha)(\sin \alpha \pi)e^{-k\alpha^2 t}$$

Since A should not be 0 (or else we get nothing!) and $e^{-k\alpha^2 t}$ is **never** 0, since forces us to have

$$\sin(\alpha\pi) = 0 \quad .$$

This means that $\alpha = 1$ or $\alpha = 2$ etc., in other words α **must** be a **positive integer** (or negative, but since $\sin(-nx) = -\sin(nx)$ and $(-n)^2 = n^2$ they don't give any thing new).

So we found **infinitely** many solutions, let's call them $u_n(x,t)$

$$u_n(x,t) = A_n(\cos nx)e^{-kn^2 t} \quad .$$

So far we have not yet imposed the **initial condition**

$$u(x,0) = f(x) \quad , \quad 0 < x < \pi \quad .$$

When we plug-in $t = 0$ in the formula for $u_n(x,t)$ we get

$$u_n(x,0) = A_n(\cos nx) \quad .$$

If you are **lucky** and $f(x)$ happens to be of that **form** i.e. $f(x)$ is some constant times $\cos x$, or some constant times $\cos 2x$, or in general some constant times $\cos nx$ for some **integer** n, then you are done, and $u_n(x,t)$ for the appropriate A_n is the **answer**. For example, if $f(x)$ happens to be

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End of story! But if $f(x)$ is something else, for example$f(x) = 1$, or $f(x) = (2x - 3\pi)x^2$ then you are out of luck, and must go on.

We found *infinitey* many solutions for the pde+boundary conditions, namely $u_n(x,t) = A_n(\cos nx)e^{-kn^2 t}$ for $n = 0, 1, 2, 3, \ldots$. (For the sake of convenience (and convention) we write $\frac{A_0}{2}$ instead of A_0). By the **principle of superposition**, the general solution is

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n(\cos nx)e^{-kn^2 t} \quad ,$$

for **any** choice of constants $A_1, A_2, A_3, \ldots$. Plugging-in $t = 0$ we get

$$u(x,0) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n (\cos nx) e^{-kn^2 \cdot 0} = \sum_{n=1}^{\infty} A_n \cos nx \quad .$$

Since $u(x,0) = f(x)$, we need to find constants $A_0, A_1, A_2, A_3, \ldots$.

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nx \quad .$$

But this is the **Fourier cosine series** from Lecture 9. So we can recover the constants A_n by the formula where

$$A_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx \quad ,$$

$$A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx \quad .$$

Problem 15.4: Using the above formula, solve the boundary value problem

$$3 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad , \quad 0 < x < \pi \quad , \quad t > 0 \quad ,$$

subject to

$$u_x(0,t) = 0 \quad , \quad u_x(\pi,t) = 0 \quad , t > 0$$

$$u(x,0) = f(x) \quad , \quad 0 < x < \pi \quad ,$$

where

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < \pi/2; \\ 1, & \text{if } \pi/2 \le x < \pi; \end{cases}$$

Solution: Here $k = 3$ so

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n (\cos nx) e^{-3n^2 t} \quad ,$$

$$A_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx = \frac{2}{\pi} \left(\int_0^{\pi/2} f(x)\, dx + \int_{\pi/2}^{\pi} f(x)\, dx \right)$$

$$= \frac{2}{\pi} \left(\int_0^{\pi/2} 0\, dx + \int_{\pi/2}^{\pi} 1\, dx \right) = \frac{2}{\pi} \int_{\pi/2}^{\pi} 1\, dx = \frac{2}{\pi} \frac{\pi}{2} = 1 \quad .$$

$$A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx = \frac{2}{\pi} \left(\int_0^{\pi/2} f(x) \cos nx \, dx + \int_{\pi/2}^{\pi} f(x) \cos nx \, dx \right)$$

$$= \frac{2}{\pi} \left(\int_0^{\pi/2} 0 \cdot \cos nx \, dx + \int_{\pi/2}^{\pi} 1 \cdot \cos nx \, dx \right) = \frac{2}{\pi} \int_{\pi/2}^{\pi} \cos nx \, dx$$

$$= \frac{2}{\pi}\left(\frac{\sin nx}{n}\Big|_{\pi/2}^{\pi}\right) = \frac{2}{n\pi}(\sin(n\pi) - \sin n(\pi/2)) = \frac{2}{n\pi}(0 - \sin n(\pi/2)) = \frac{-2\sin n(\pi/2)}{\pi n}$$

So

$$u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-2\sin(n\pi/2)}{\pi n}(\cos nx)e^{-3n^2 t} \quad .$$

First Ans. to 15.4:

$$u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-2\sin(n\pi/2)}{\pi n}(\cos nx)e^{-3n^2 t} \quad .$$

Since $\sin n\pi/2 = 0$ when n is even and $\sin(2k+1)\pi/2 = (-1)^k$, we can simplify the above and write:

$$\frac{1}{2} + \sum_{k=0}^{\infty} \frac{-2(-1)^k}{\pi(2k+1)} \cos(2k+1)x = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2(-1)^{k+1}}{\pi(2k+1)}(\cos(2k+1)x)e^{-3(2k+1)^2 t} \quad .$$

Better Ans. to 15.4: $\frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1}(\cos(2k+1)x)e^{-3(2k+1)^2 t}$.