Dr. Z.'s Calc4 Lecture 9 Handout: Solutions of Linear Homogeneous Equations and the Wronskian

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The general form of a **homogeneous second order linear diff.eq.** is

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = 0 \quad ,$$

where, in general, the **coefficients**, $a(t), b(t), c(t)$ are **functions** of t (not just *constants*).

If the right side is **not** zero, then we have an **inhomogeneous second order linear diff.eq.** whose format is:

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = d(t) \quad ,$$

where $d(t)$ is yet another function of t.

By dividing by $a(t)$, and putting $p(t) = \frac{b(t)}{a(t)}$, $q(t) = \frac{c(t)}{a(t)}$, $g(t) = \frac{d(t)}{a(t)}$, it may be written

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t) \quad ,$$

where $p(t)$, $q(t)$, and $g(t)$ are other functions of t.

Existence and Uniqueness Theorem for Second-Order Linear Diff.eqs.

If the coefficient-functions $p(t), q(t), g(t)$ are **continuous** (i.e. "well-behaved", they have no breaks and do not blow up) in an open interval I containing the number t_0, then we are **guaranteed** that there **exists** a **unique** solution $y(t)$ to the inital value diff.eq.

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t) \quad , \quad y(t_0) = y_0 \quad , \quad y'(t_0) = z_0 \quad ,$$

for *any* numbers y_0, z_0, that makes sense through the open interval I.

In other words, if the functions $p(t), q(t), g(t)$ do not blow up, and have no surprises in I, and you want a function $y(t)$ that satisfies the diff.eq. and its value at t_0 is y_0, and the value of its rate-of-change at t_0 is z_0, then you are *promised* that there is such a function. Not only that, there is only **one** such function!

Problem 9.1: For each of the following diff.eq. inital value problems, and intervals, decide whether the theorem promises you that there is a unique solution.

a. $(t^2+1)\,y''(t) + \sin t\, y'(t) + (t^3+1)\,y(t) = e^t \quad , \quad y(0) = 1 \quad , \quad y'(0) = 3 \quad ; \quad -10 < t < 10$

b. $t^2\,y''(t) + \sin t\, y'(t) + (t^3+1)\,y(t) = e^t \quad , \quad y(0) = 1 \quad , \quad y'(0) = 3 \quad ; \quad -10 < t < 10$

c. $y''(t) + \tan t\, y'(t) + (t^3+1)\,y(t) = e^{t^2} \quad , \quad y(0) = 1 \quad , \quad y'(0) = 3 \quad ; \quad 0 < t < \frac{2\pi}{3}$

d. $(t-5)(t-3)\,y''(t) + y'(t) + y(t) = e^{t^2} \quad , \quad y(0) = 1 \quad , \quad y'(0) = 3 \quad ; \quad -2 < t < 2$

Solution to 9.1a: We first **divide** by the coefficient of $y''(t)$, t^2+1, getting the equivalent diff.eq. (inital value problem)

a.' $y''(t) + \frac{\sin t}{t^2+1} y'(t) + \frac{t^3+1}{t^2+1} y(t) = \frac{e^t}{t^2+1}$, $y(0)=1$, $y'(0)=3$; $-10 < t < 10$

So $p(t) = \frac{\sin t}{t^2+1}$, $q(t) = \frac{t^3+1}{t^2+1}$, and $g(t) = \frac{e^t}{t^2+1}$. The numerators are all nice (continuous) functions, and the denominator in these, t^2+1 never vanishes in the interval $(-10, 10)$, so everything is nice, and the theorem promises a unique solution.

Solution to 9.1b: We first **divide** by the coefficient of $y''(t)$, t^2, getting the equivalent diff.eq. (inital value problem)

b.' $y''(t) + \frac{\sin t}{t^2} y'(t) + \frac{t^3+1}{t^2} y(t) = \frac{e^t}{t^2}$, $y(0)=1$, $y'(0)=3$; $-10 < t < 10$

So $p(t) = \frac{\sin t}{t^2}$, $q(t) = \frac{t^3+1}{t^2}$, and $g(t) = \frac{e^t}{t^2}$. The numerators are all nice (continuous) functions, **but** the denominator in these, t^2 vanishes at $t=0$, and $t=0$ happens to lie in the given interval, so (at least one of, but in this case all of) $p(t), q(t), g(t)$ blow up, so are **not continuous**, and there is no guarantee.

Solution of 9.1c: $\tan t$ blows up at $t = \frac{\pi}{2}$ and since it lies in the given interval $0 < t < \frac{2\pi}{3}$, it is not continuous and there is **no guarantee**.

Solution of 9.1d: After dividing by $(t-3)(t-5)$ we see that the coefficient functions blow-up at $t=3$ and $t=5$. But neither trouble spots lie in the interval $-2 < t < 2$ so we are safe and we are **guaranteed a unique solution**.

Ans. to 9.1: **a** and **d** are guaranteed to have a unique solution, but **b** and **c** are not.

Problem 9.2: Find the largest open interval for which the solution to

$(t-1)(t+3)\, y''(t) + \sin t y'(t) + \cos t y(t) = t$, $y(0)=1$, $y'(0)=3$

has a unique solution.

Solution to 9.2: After dividing by the coefficient of $y''(t)$, $(t-1)(t+3)$, we get the initial value problem

$y''(t) + \frac{\sin t}{(t-1)(t+3)} y'(t) + \frac{\cos t}{(t-1)(t+3)} y(t) = \frac{t}{(t-1)(t+3)}$, $y(0)=1$, $y'(0)=3$

The coefficients blow-up at $t=-3$ and $t=1$. Our interval should contain $t_0=0$, so the largest interval is $-3 < t < 1$.

Ans. to 9.2: The largest interval is $-3 < t < 1$.

We already know about the amazing **Principle of Superposition** (only valid for **homogeneous** diff.eqs.!)

If $y_1(t)$ and $y_2(t)$ are two solutions of the second-order homog.linear diff.eq.

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0 \quad ,$$

then we can come up with (doubly!) infinite family of functions that also satisfy the same diff.e.q

$$c_1 y_1(t) + c_2 y_2(t)$$

where c_1, c_2 are *arbitrary constants* free to range from $-\infty$ to ∞.

Usually, every solution is of that form. To find out whether this is the case we need the imporant concept of *Wronskian*.

Important Definition (The Wronskian): The *Wronskian* of two functions $f(t), g(t)$ is the brand-new function

$$W(f(t), g(t)) = f(t)g'(t) - f'(t)g(t) \quad .$$

Problem 9.3: Find the Wronskian, $W(f(t), g(t))$ of the following pair of functions.

a: $f(t) = t^3 \quad , \quad g(t) = t^2$

b: $f(t) = e^t \quad , \quad g(t) = \sin t$

c: $f(t) = \sin t \quad , \quad g(t) = \cos t$

Solution to 9.3:

a.: $W(t^3, t^2) = t^3(t^2)' - (t^3)'(t^2) = t^3(2t) - (3t^2)(t^2) = 2t^4 - 3t^4 = -t^4$

b.: $W(e^t, \sin t) = e^t(\sin t)' - (e^t)'(\sin t) = e^t \cos t - e^t \sin t = e^t(\cos t - \sin t)$

c.: $W(\sin t, \cos t) = \sin t(\cos t)' - (\sin t)'(\cos t) = -\sin^2 t - \cos^2 t = -1$

Important Theorem: Suppose that $y_1(t)$ and $y_2(t)$ are two specific solutions of the second-order linear diff.eq.

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0 \quad ,$$

Then **every** other solution can be written as $c_1 y_1(t) + c_2 y_2(t)$ for *some* **constants** c_1, c_2 if and only if the **Wronskian** , $W(y_1, y_2)$ is **not identically zero**.

Note: it is OK if the Wronskian has some zero-point(s). For example if $y_1(t) = t$, $y_2(t) = t^2$ then $W(t, t^2) = t(t^2)' - (t)'(t^2) = t^2$ and it is 0 when $t = 0$, but t and t^2 are still an OK **fundamental set**. But if $y_1 = t^2, y_2 = 2t^2$ then $W(t^2, 2t^2) = 0$ (always!), so these are not OK.

Most diff.eq. are impossible to solve (or very hard), so given a linear second-order diff.eq.

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0 \quad ,$$

and you want to find the Wronskian of any two of its solutions $y_1(t), y_2(t)$, then it is either very hard, or impossible, to compute the Wronskian of $y_1(t), y_2(t)$ **directly**. But thanks to **Nils Abel**, we don't! We can figure it out (up to a constant multiple c) from the diff.eq. itself **without** bothering to solve it!

Abel's Theorem

The Wronskian of any two solutions $y_1(t), y_2(t)$ of the above linear homog. diff.eq. is always given by

$$c\,exp\left[-\int p(t)\,dt\right] \quad ,$$

where c is a **constant**.

Note: If $y_1(t)$ and $y_2(t)$ are constant multiples of each other then c happens to be 0.

Problem 9.4: Find the Wronskian (up to a constant in front) of any two solutions of the following diff.eq.

$$y''(t) + t^2\,y'(t) + e^t\,y(t) = 0 \quad .$$

Sol. to 9.4: Here $p(t) = t^2$ (note: $q(t)$ is irrelevant for this problem!), we have

$$W(y_1, y_2) = c\,exp\left[-\int t^2\,dt\right] = c\,exp[-\frac{t^3}{3}] = c\,e^{-t^3/3} \quad .$$

Ans. to 9.4: The Wronskian is $c\,e^{-t^3/3}$ for some constant c.