Dr. Z.'s Calc4 Lecture 8 Handout:
Homogeneous Second-Order Differential Equations With Constant Coefficients

By Doron Zeilberger

The general form of a *general* (usually non-linear) **second-order** diff.eq. is

$$y''(t) = F(t, y(t), y'(t)) \quad .$$

Most diff. eq. can't be solved "analytically", i.e. via a closed-form expression featuring polynomials and the familiar exponential and trig functions, and one can only find numerical approximations. But for **special classes** of 2nd order diff.eqs. there are **special methods**.

An important class (still difficult) is the family of **linear second-order differential equations**. Its format is

$$P(t)y''(t) + Q(t)y'(t) + R(t)y(t) = S(t) \quad ,$$

where $P(t), Q(t), R(t), S(t)$ are some functions of t. For example, the following are linear diff.eqs.

$$t\,y''(t) + \sin t\,y'(t) + e^t y(t) = t^3 \quad ,$$

but the following is **not** linear

$$t\,y''(t) + \sin t\,y'(t) + e^t y(t)^2 = t^3 \quad ,$$

since $y(t)$ appears squared. Neither is

$$t\,y''(t) + \sin t\,y'(t)y(t) = t^3 \quad ,$$

since $y(t)$ and $y'(t)$ are multiplied by each other. To qualify as **linear** each of $y''(t)$, $y'(t)$ and $y(t)$ must be **alone**.

If the right-side, $S(t)$ happens to be 0, i.e. we have a linear 2nd order diff.eq. of the format

$$P(t)y''(t) + Q(t)y'(t) + R(t)y(t) = 0 \quad ,$$

then such a linear diff.eq. is called **homogeneous**.

Homogeneous diff.eqs. have two *amazing* properties.

First amazing property: If $\phi(t)$ is a solution of a *homogeneous* linear (2nd-order) diff. eq. $P(t)y''(t) + Q(t)y'(t) + R(t)y(t) = 0$, then so is any **constant** multiple, $c\phi(t)$ (c a number).

Second amazing property: If $\phi_1(t)$ and $\phi_2(t)$ are both solutions of a *homogeneous* linear (2nd-order) diff. eq. $P(t)y''(t) + Q(t)y'(t) + R(t)y(t) = 0$, then so is their sum: $\phi_1(t) + \phi_2(t)$.

The proofs are easy (you do it!).

Even for the special case of homog. 2nd-order linear diff. eqs. it is usually not possible to solve them (of course you can declare the solution to be a new function, and name it after yourself). But if the *coefficients* $P(t), Q(t), R(t)$ happen to be **constant** functions (do not depend on t), then one can always find explicit solutions.

A general **homogeneous 2nd-order linear diff.eq. with constant coefficients** has the format

$$ay''(t) + by'(t) + cy(t) = 0 \quad ,$$

where a, b, c are (fixed!) **numbers**. You try your luck with a solution of the form

$$y(t) = e^{rt} \quad ,$$

and plug it into the diff.eq. We have $y(t) = e^{rt}$, $y'(t) = re^{rt}$, $y''(t) = r^2e^{rt}$

and get

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0 \quad .$$

Factoring out e^{rt}:

$$(ar^2 + br + c)e^{rt} = 0 \quad .$$

Since e^{rt} is never zero, we can divide by it (or if you wish, multiply by e^{-rt}) and get

$$ar^2 + br + c = 0 \quad .$$

This is no longer a differential equation, but rather a simple (quadratic) **algebraic equation**.

If it has two distinct real roots (we will talk about multiple and complex roots later), let's call them r_1, r_2, then it means that we found **two** *independent* solutions e^{r_1t} and e^{r_2t}. By the first amazing property any constant multiples (with [usually] different constants) $c_1e^{r_1t}$ and $c_2e^{r_2t}$ are also solutions. By the second amazing property for homogeneous linear diff.eqs so it the sum, so

$$y(t) = c_1e^{r_1t} + c_2e^{r_2t} \quad ,$$

are **all** solutions of the diff.eq. $ay''(t) + by'(t) + cy(t) = 0$. It is possible to prove that every solution is of that form, so we have

Important Theorem

The **general solution** of the **homogeneous linear second-order differential equation with constant coefficients**

$$ay''(t) + by'(t) + cy(t) = 0 \quad ,$$

is

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \quad ,$$

where r_1, r_2 are the two distinct roots of the quadratic equation (assuming that $b^2 - 4ac > 0$)

$$ar^2 + br + c = 0 \quad ,$$

and C_1, C_2 are *arbitrary constants*.

Note: The quadratic equation $ar^2 + br + c = 0$ is called the **characteristic equation**.

Problem 8.1 Find the general solution of the following diff. eq.

$$y'' - 3y' + 2y = 0 \quad .$$

Step 1: Write down the **characteristic equation** by replacing y'' by r^2, y' by r and y by 1

$$r^2 - 3r + 2 = 0 \quad .$$

Step 2: Solve the quadratic equation, either using $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$, or, if possible, by factorization.

$$(r-1)(r-2) = 0 \quad ,$$

so we have two distinct roots, $r_1 = 1$, $r_2 = 2$.

Step 3: Write down the general solution

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \quad .$$

In this problem the two roots are $r_1 = 1, r_2 = 2$, so we have

$$y(t) = C_1 e^{t} + C_2 e^{2t} \quad ,$$

Ans. to 8.1: $y(t) = C_1 e^t + C_2 e^{2t}$.

Problem 8.2 Find the general solution of the following diff. eq.

$$y'' - 3y' + y = 0 \quad .$$

Step 1: Write down the **characteristic equation** by replacing y'' by r^2, y' by r and y by 1

$$r^2 - 3r + 1 = 0 \quad .$$

Step 2: Solve the quadratic equation, either using $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, or, if possible, by factorization. Now it is not possible to factorize, so we have

$$r_1, r_2 = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(1)}}{2 \cdot 1} = \frac{3 \pm \sqrt{5}}{2} = \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2} \quad .$$

so we have two distinct roots, $r_1 = \frac{3}{2} + \frac{\sqrt{5}}{2}$, $r_2 = \frac{3}{2} - \frac{\sqrt{5}}{2}$,

Step 3: Write down the general solution

$$y(t) = C_1 e^{(\frac{3}{2}+\frac{\sqrt{5}}{2})t} + C_2 e^{(\frac{3}{2}-\frac{\sqrt{5}}{2})t} = e^{\frac{3}{2}t}(C_1 e^{\frac{\sqrt{5}}{2}t} + C_2 e^{-\frac{\sqrt{5}}{2}t})$$

Ans. to 8.2: $y(t) = e^{\frac{3}{2}t}(C_1 e^{\frac{\sqrt{5}}{2}t} + C_2 e^{-\frac{\sqrt{5}}{2}t})$.

The **initial value problems** for second-order diff.eqs. have **two** data $y(0), y'(0)$ (or more generally $y(t_0), y'(t_0)$).

Problem 8.3 Find the solution of the following initial value 2nd-order diff. eq.

$$y'' - 3y' + 2y = 0 \quad , \quad y(0) = 2 \quad , \quad y'(0) = 3 \quad .$$

Step 1-3: Find the general solution, like in Problem 8.1.

$$y(t) = C_1 e^t + C_2 e^{2t} \quad .$$

Step 4: Find $y'(t)$

$$y'(t) = C_1 e^t + 2C_2 e^{2t} \quad .$$

and plug-in $t = t_0$ into the general $y(t)$ and $y'(t)$.

$$y(0) = C_1 e^0 + C_2 e^0 = C_1 + C_2 \quad ,$$

$$y'(0) = C_1 e^0 + 2C_2 e^0 = C_1 + 2C_2 \quad .$$

Step 5: Implement the initial conditions:

$$2 = C_1 + C_2 \quad ,$$

$$3 = C_1 + 2C_2 \quad .$$

Step 6: Solve the system of two equations and two unknowns getting C_1, C_2, certain specific numbers. Subtracting the second eq. from the first gives $C_2 = 1$, plugging into the first (or the second) gives $C_1 = 1$.

Step 7: Go back to the general solution above and substitute the C_1, C_2.

$$y(t) = 1 \cdot e^t + 1 \cdot e^{2t} = e^t + e^{2t} \quad .$$

Ans. to 8.3: $y(t) = e^t + e^{2t}$.

Problem 8.4 Find the solution of the following inital value diff. eq.

$$y'' - y = 0 \quad , \quad y(-2) = 1 \quad , \quad y'(-2) = -1 \quad .$$

Steps 1-3: We first find the general solution, featuring C_1, C_2. (i) $r^2 - 1 = 0$ (ii) $r = -1, r = 1$ (iii) $y(t) = C_1 e^{-t} + C_2 e^t$

Step 4: Find $y'(t)$

$$y'(t) = -C_1 e^{-t} + C_2 e^t$$

and plug-in $t = t_0$ into the general $y(t)$ and $y'(t)$.

$$y(-2) = C_1 e^2 + C_2 e^{-2}$$

$$y'(-2) = -C_1 e^2 + C_2 e^{-2}$$

Step 5: Implement the initial conditions:

$$1 = C_1 e^2 + C_2 e^{-2}$$

$$-1 = -C_1 e^2 + C_2 e^{-2}$$

Step 6: Solve the system of two equations and two unknowns getting C_1, C_2, certain specific numbers.

Adding them we get: $2C_2 e^{-2} = 0$ so $C_2 = 0$. Subtracting (or plugging-in) gives $2 = 2C_1 e^2$ so $C_1 = e^{-2}$ and $C_2 = 0$.

Step 7: Go back to the general solution above and substitute the C_1, C_2.

$$y(t) = C_1 e^{-t} + C_2 e^t = e^{-2} e^{-t} + 0 e^t = e^{-t-2} \quad .$$

Ans. to 8.4: $y(t) = e^{-t-2}$.