Dr. Z.'s Calc4 Lecture 6 Handout

By Doron Zeilberger

[Version of Sept. 23, 2013, correcting a typo, thanks to Jonathan Chang]

Some first order diff.eqs. are neither linear nor separable, but you can still solve them exactly, at least implicitly, (i.e. find some relation $F(x, y(x)) = Constant$).

Let's see how to "cook" such a lucky first-order diff.eq., by going **backwards**. Suppose that some function $y(x)$ is given implicitly by some relation

$$F(x, y(x)) = 0 \quad .$$

Let's differentiate with respect to x using the multivariable **chain rule**.

$$0 = \frac{\partial}{\partial x} F(x, y(x)) = F_x(x, y(x)) \frac{\partial x}{\partial x} + F_y(x, y(x)) \frac{\partial y(x)}{\partial x}$$

So:

$$F_x(x, y(x)) + F_y(x, y(x)) y' = 0 \quad .$$

This has the format, for some functions $M(x, y)$ and $N(x, y)$,

$$M(x, y) + N(x, y) y' = 0 \quad .$$

Of course, not every first-order diff.eq. can be written like this, only those for which there is a **magical** $F(x, y)$ for which

$$M(x, y) = F_x(x, y) \quad , \quad N(x, y) = F_y(x, y) \quad .$$

How can we tell, beforehand, whether there is such a magic $F(x, y)$? If there is one, then we **must** have

$$M_y(x, y) = N_x(x, y) \quad ,$$

since the left side is $F_{xy}(x, y)$ and the right side is $F_{yx}(x, y)$ and we know from Calc3 that these are equal!

So we have the following

Definition: A first-order diff.eq. that is written in the form

$$M(x, y) + N(x, y)\, y' = 0 \quad ,$$

for some functions $M(x, y), N(x, y)$ of (x, y) is **exact** if

$$M_y(x, y) = N_x(x, y) \quad .$$

Problem 6.1 For the following first-order diff.eqs., decide whether or not they are exact.

a.

$$(3x^2y + 2xy^5 + 5y) + (x^3 + 5x^2y^4 + 5x)\, y' = 0$$

b.

$$(3x^2y + 2xy^5 + 6y) + (x^3 + 5x^2y^4 + 5x)\, y' = 0$$

c.

$$\cos(x + y)e^y + (\cos(x + y) + \sin(x + y))e^y\, y' = 0$$

d.

$$(2xe^{x^2+y^3} + y) + (3y^2e^{x^2+y^3} + x)\, y' = 0$$

e.

$$(2xe^{x^2+y^3} + xy) + (3y^2e^{x^2+y^3} + 3)\, y' = 0$$

Solutions to 6.1

a: $M = 3x^2y + 2xy^5 + 5y$, $N = x^3 + 5x^2y^4 + 5x$. So $M_y = 3x^2 + 10xy^4 + 5$, $N_x = 3x^2 + 10xy^4 + 5$. Since $M_y = N_x$ this diff.eq. is **exact**.

b: $M = 3x^2y + 2xy^5 + 6y$, $N = x^3 + 5x^2y^4 + 5x$. So $M_y = 3x^2 + 10xy^4 + 6$, $N_x = 3x^2 + 10xy^4 + 5$. Since $M_y \neq N_x$ this diff.eq. is **NOT exact**.

c: $M = \cos(x + y)e^y$, $N = (\cos(x + y) + \sin(x + y))e^y$. $M_y = (-\sin(x + y) + \cos(x + y))e^y$, $N_x = (-\sin(x + y) + \cos(x + y))e^y$. Since $M_y = N_x$ this diff.eq. is **exact**.

d. $M = 2xe^{x^2+y^3} + y$, $N = 3y^2e^{x^2+y^3} + x$. So $M_y = 2xe^{x^2+y^3}(3y^2) + 1 = 6xy^2e^{x^2+y^3} + 1$ and $N_x = 3y^2e^{x^2+y^3}(2x) + 1 = 6xy^2e^{x^2+y^3} + 1$. Since $M_y = N_x$ this diff.eq. is **exact**.

e. $M = 2xe^{x^2+y^3} + xy$, $N = 3y^2e^{x^2+y^3} + 3$. So $M_y = 2xe^{x^2+y^3}(3y^2) + x = 6xy^2e^{x^2+y^3} + x$ and $N_x = 3y^2e^{x^2+y^3}(2x) = 6xy^2e^{x^2+y^3}$. Since $M_y \neq N_x$ this diff.eq. is **not exact**.

Ans. to Problem 6.1: **a,c,d** are exact diff.eqs. and **b,e** are not exact.

Suppose that we found out that our diff.eq.

$$M(x, y) + N(x, y)y' = 0$$

is exact, how do find $F(x, y)$? Once we will find this magic $F(x, y)$, we know that

$$\frac{\partial}{\partial x}F(x, y) = 0 \quad ,$$

so the solution would be (implicitly!)

$$F(x,y) = C \quad ,$$

for a general constant C. If there is an initial condition given, $y(x_0) = y_0$ we find C by plugging-in. $F(x_0, y_0)$

How do we find $F(x,y)$?

Since

$$F_x(x,y) = M(x,y) \quad ,$$

we integrate both sides w.r.t. to x getting

$$F(x,y) = \int M(x,y)\, dx + SomeThingThatDoesNotDependOnx \quad ,$$

But $SomeThingThatDoesNotDependOnx$ means a function of y only, so we call it $\phi(y)$.

Now use the fact that

$$F_y(x,y) = N(x,y) \quad ,$$

and this leaves us, after the algebra, with some expression for $\phi'(y)$, and to get $\phi(y)$ we integrate w.r.t. to y.

Problem 6.2: Decide whether the following diff.eq. is exact. If it is, then solve it.

$$(3x^2y + 2xy^5 + 5y) + (x^3 + 5x^2y^4 + 5x + 2\sin 2y)y' = 0 \quad .$$

Solution of 6.2

$M = 3x^2y + 2xy^5 + 5y$, $N = x^3 + 5x^2y^4 + 5x + 2\sin 2y$. So $M_y = 3x^2 + 10xy^4 + 5$, $N_x = 3x^2 + 10xy^4 + 5$. Since $M_y = N_x$ this diff.eq. is **exact**. Now

$$F_x = (3x^2y + 2xy^5 + 5y) \quad ,$$

So

$$F = \int (3x^2y + 2xy^5 + 5y)\, dx = x^3y + x^2y^5 + 5yx + \phi(y) \quad ,$$

where $\phi(y)$ is **to be determined** (using the other clue, that $F_y = N$).

Putting the tentative $F(x,y)$ in

$$F_y = N \quad ,$$

we get

$$\frac{\partial}{\partial y}(x^3y + x^2y^5 + 5yx + \phi(y)) = x^3 + 5x^2y^4 + 5x + 2\sin 2y \quad ,$$

So

$$x^3 + 5x^2y^4 + 5x + \phi'(y) = x^3 + 5x^2y^4 + 5x + 2\sin 2y \quad .$$

Simplifying, we get

$$\phi'(y) = 2\sin 2y \quad .$$

Warning: if at this stage the right side would feature an expression that contains x in it (in addition to y), it means that you messed up! Either the diff.eq. is not exact, or you made another mistake. **DO NOT DARE TO CONTINUE**. It is better to admit error than to continue and get utter nonsense.

In this case everything is OK $\phi'(y)$ **only depends** on y, and we can integrate w.r.t. to y:

$$\phi(y) = \int 2\sin 2y\, dy = -\cos 2y + C \quad .$$

Going back to the tentative F above, we have

$$F(x, y) = x^3y + x^2y^5 + 5yx - \cos 2y.$$

and the **Final Answer is**

$$x^3y + x^2y^5 + 5yx - \cos 2y = C \quad ,$$

where C is an *arbitrary constant.*

Ans. to 6.2: $x^3y + x^2y^5 + 5yx - \cos 2y = C$ (where C is an arbitrary constant).

Problem 6.3: Solve the initial value problem

$$(3x^2y + 2xy^5 + 5y) + (x^3 + 5x^2y^4 + 5x + 2\sin 2y)y' = 0 \quad , \quad y(1) = \pi/4.$$

Solution of 6.3. The diff.eq. is the same as for Problem 6.2, so we do everything as before. Since we have an initial condition, we have an extra step. Plug-in $x = 1$, $y = \pi/4$ into the general (implicit) solution.

$$x^3y+x^2y^5+5yx-\cos 2y = 1^3\cdot(\pi/4)+1^2\cdot(\pi/4)^5+5(\pi/4)(1)-\cos(\pi/2) = 6\pi/4+(\pi/4)^5-0 = \frac{3}{2}\pi+\frac{\pi^5}{1024} \quad .$$

So the final answer is

$$x^3y + x^2y^5 + 5yx - \cos 2y = \frac{3}{2}\pi + \frac{\pi^5}{1024} \quad .$$

Ans. to 6.3: $x^3y + x^2y^5 + 5yx - \cos 2y = \frac{3}{2}\pi + \frac{\pi^5}{1024}$.

Integrating factor

Of course if we have a diff. eq.

$$M(x, y) + N(x, y)y' = 0$$

then automatically, for any $\mu(x, y)$, we have also

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)y' = 0 \quad .$$

So if our original diff.eq. is not exact, there is hope that for some magical $\mu(x, y)$ the new diff. eq. would be exact.

I don't think that there is a general way of finding such magical $\mu(x, y)$ (efficiently), but in the special case that there is a $\mu(x)$ that **only** depends on x, then it must satisfy the simple diff.eq.

$$\frac{\mu'(x)}{\mu(x)} = \frac{M_y - N_x}{N} \quad ,$$

if you are lucky and the right side to **only** depend on x. (There is something analogous if μ only depends on y.)

Problem 6.4: By finding an appropriate integrating factor (a function of x) solve the diff.eq

$$(3xy^2 + 2y^3) + (2x^2y + 3xy^2)y' = 0 \quad .$$

Sol. of 6.4: $M = 3xy^2 + 2y^3$, $N = 2x^2y + 3xy^2$. $M_y - N_x = (6xy + 6y^2) - (4xy + 3y^2) = 2xy + 3y^2$. Since $M_y - N_x \neq 0$ the original diff.eq. is **not exact**. But

$$\frac{M_y - N_x}{N} = \frac{2xy + 3y^2}{2x^2y + 3xy^2} = \frac{y(2x + 3y)}{xy(2x + 3y)} = \frac{1}{x} \quad .$$

Yea, it only depends on x! (if it depends on both x and y, the method fails).

So

$$\frac{\mu'(x)}{\mu(x)} = \frac{1}{x} \quad ,$$

So

$$\frac{d\mu}{\mu} = \frac{dx}{x} \quad .$$

Integrating

$$\ln \mu = \ln x + C \quad .$$

Exponentiating:

$$\mu = Cx \quad ,$$

(but we can take $C = 1$). So the magic **integrating factor** is $\mu(x) = x$. Multiplying the original diff.eq. by x we get the equivalent diff.eq.:

$$(3x^2y^2 + 2xy^3) + (2x^3y + 3x^2y^2)y' = 0 \quad .$$

Now, $M = 3x^2y^2 + 2xy^3$, $N = 2x^3y + 3x^2y^2$. $M_y - N_x = (6x^2y + 6xy^2) - (6x^2y + 6xy^2) = 0$, so the diff. eq. is **indeed exact**. Next,

$$F = \int M\,dx = \int (3x^2y^2 + 2xy^3)\,dx = x^3y^2 + x^2y^3 + \phi(y)$$

Next

$$F_y = N$$

means

$$2x^3y + 3x^2y^2 + \phi'(y) = 2x^3y + 3x^2y^2$$

So

$$\phi'(y) = 0 \quad ,$$

and $\phi(y) = C$, and the (general) solution is

$$x^3y^2 + x^2y^3 = C \quad .$$

Ans. to 6.4: $x^3y^2 + x^2y^3 = C$.