Dr. Z.'s Calc4 Lecture 4 Handout: Existence and Uniqueness of First-Order Diff.Eqs.

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There are two **important** theorems that tells when you are guaranteed to have solutions to first-order diff.eqs. and whether the solution is unique. The first one applies to *linear* (first-order) diff.eq. and the second applies to *non-linear* diff.eq.

Existence and Uniqueness Theorem for LINEAR first-order Diff.Eq.

If the functions $p(t)$ and $g(t)$ are continuous (do not blow up or have breaks, in other words are well-behaved) on an open interval $\alpha < t < \beta$ containing the number t_0, and y_0 is *any* number, then there exists a **unique** function $y = \phi(t)$ that is a solution of the first-order linear diff.eq.

$$y'(t) + p(t)y(t) = g(t) \quad ,$$

for all t in the interval $\alpha < t < \beta$, and satisfies the initial condition $y(t_0) = y_0$.

Problem 4.1: Without actually solving the diff.eq.s decide whether the following initial value problems have solutions and whether there are unique, for the specified intervals.

a. $y'(t) + t^3\, y(t) = \sin t$, $-10 < t < 10$, $y(8) = 1000$

b. $y'(t) + \frac{1}{t}\, y(t) = \sin t$, $-10 < t < 10$, $y(1) = 10$

c. $y'(t) + \frac{1}{t-2}\, y(t) = \sqrt{t}$, $3 < t < 5$, $y(4) = 10$

d. $y'(t) + \frac{1}{(t-3)^3}\, y(t) = e^{-t^2}$, $2 < t < 5$, $y(4) = 10$

e. $y'(t) + \cos t\, y(t) = \frac{1}{t-3}$, $4 < t < 6$, $y(5) = -100$

f. $y'(t) + \cos t\, y(t) = \frac{1}{t+300}$, $-\infty < t < \infty$, $y(0) = 5$.

Sol. to Problem 4.1

For **a**, both t^3 and $\sin t$ are *always* nice (never blow up, or have breaks) so we are guaranteed a unique solution for **any** interval, and any initial condition.

For **b**, $\sin t$ is nice **but** $\frac{1}{t}$ blows up at $t = 0$ ($1/0$ is nonsense!), since $t = 0$ happens to be **inside** the interval $-10 < t < 10$, the theorem **does NOT** apply, and no conclusion can be drawn!

For **c**, $\sqrt{t}$ is nice for $t > 0$, and since our interval is $3 < t < 5$ there is no problem. $\frac{1}{t-2}$ does blow-up at $t = 2$, but since it does **not** belong to our interval, everything is OK! So we are guaranteed that there is a (unique!) solution.

For **d**, e^{-t^2} is always nice, but $\frac{1}{(t-3)^3}$ blows up at $t = 3$. Unfortunately, his happens to lie inside our interval $2 < t < 5$, so no conclusion can be drawn!

For **e**, $\cos t$ is always nice, but $\frac{1}{t-3}$ blows up at $t = 3$. Luckily this trouble-spot happens to lie **outside** our interval $4 < t < 6$, so we are *guaranteed* a unique solution.

For **f**, $\cos t$ is always nice, but $\frac{1}{t+300}$ blows up at $t = -300$. Since this happens to lie **inside** our interval $-\infty < t < \infty$, so no conclusion can be drawn.

Problem 4.2: Find the maximal open intervals for which the following first-order diff.eq. is guaranteed to have a unique solution.

$$y'(t) + \frac{t^2}{(t-1)(t-4)}\, y(t) = \frac{t^5}{t-2} \quad .$$

Solution of 4.2: Here $p(t) = \frac{t^2}{(t-1)(t-4)}, g(t) = \frac{t^5}{t-2}$. $p(t)$ is well-behaved **except** at $t = 1$ and $t = 4$, where it blows up, and $g(t)$ is well-behaved except at $t = 2$. So the "trouble spots" are $t = 1, t = 2, t = 4$. An open interval that contains any of these bad spots is no good, so the maximal "good" intervals are

$$-\infty < t < 1 \quad , \quad 1 < t < 2 \quad , \quad 2 < t < 4 \quad , 4 < t < \infty \quad .$$

The second important theorem is for existence and uniqueness of a **non-linear** diff.eq. (of course, the theorem applies to linear diff.eqs. as well.

Existence and Uniqueness Theorem for GENERAL first-order Diff.Eq.

If the function $f(t, y)$ of the **two** variables t and y, as well as $f_y(t, y)$ (aka $\frac{\partial f}{\partial y}(t, y)$) are continuous in some rectangle in the t, y plane,

$$\alpha < t < \beta \quad , \quad \gamma < y < \delta \quad ,$$

that contains the point (t_0, y_0), then you are **guaranteed** to have a (unique!) solution $y(t) = \phi(t)$ satisfying the first-order diff.eq. initial value problem

$$y'(t) = f(t, y) \quad y(t_0) = y_0 \quad ,$$

that is valid for **some** interval $t_0 - h_0 < t < t_0 + h_0$ contained in $\alpha < t < \beta$.

Comments 1. This is still applicable (but not as strong as the first theorem) for linear diff.eq. $y'(t) + p(t)\, y(t) = g(t)$. Simply take $f(t, y) = g(t) - p(t)\, y(t)$.

2. Note that there is no guarantee that the solution is valid (i.e. does not blow up) for the entire $\alpha < t < \beta$, all we are promised that one can find *some* interval (possibly tiny!) around $t = t_0$ where the solution makes sense.

Problem 4.3: For which of the following initial value problems are we guaranteed to have a unique solution, $y(t)$ on *some* interval of the t-line, around the initial time?

a. $y' = \frac{1}{2-y-t}$, $y(1) = 3$

b. $y' = \frac{1}{3-y-t}$, $y(1) = 2$

c. $y' = \frac{1}{1+y^2+t^2}$, $y(0) = 30000$

d. $y' = t + y^{\frac{1}{5}}$, $y(0) = 0$.

Solution of 4.3a: Here $t_0 = 1, y_0 = 3$, $f(t,y) = \frac{1}{2-y-t}$.Also $f_y(t,y) = \frac{1}{(2-y-t)^2}$. Since $f(t_0,y_0) = \frac{1}{2-3-1} = -\frac{1}{2}$ does **not** blow-up (since the denominator is **not** 0), and ditto for $f_y(t_0,y_0) = \frac{1}{(2-3-1)^2} = \frac{1}{4}$, we can find some rectangle around $(1,3)$ where both $f(t,y)$ and $f_y(t,y)$ are well-behaved. So for *some* interval containing $t = 1$ (we do not know how long, beforehand) there is a function $\phi(t)$ such that we have *both* $\phi'(t) = \frac{1}{2-\phi(t)-t}$, *and* $\phi(1) = 3$.

Solution of 4.3b: Here $t_0 = 1, y_0 = 2$, $f(t,y) = \frac{1}{3-t-y}$. Since $f(t_0,y_0) = \frac{1}{3-1-2} = \frac{1}{0}$ blows-up there is no solution!

Solution of 4.3c: Here $t_0 = 0, y_0 = 30000$, $f(t,y) = \frac{1}{1+y^2+t^2}$. Also $f_y(t,y) = -\frac{2y}{(1+y^2+t^2)^2}$. Since $f(0,30000)$ is **not** 0, and neither is and $f_y(0,30000)$, we can find *some* rectangle around $(0,30000)$ where both $f(t,y)$ and $f_y(t,y)$ are well-behaved. So for *some* interval containing $t = 0$ we are guaranteed a unique solution.

Solution of 4.3d: Here $t_0 = 0, y_0 = 0$, $f(t,y) = t + y^{\frac{1}{5}}$.Also $f_y(t,y) = \frac{1}{5}y^{-\frac{4}{5}}$. $f(t,y)$ is OK near $(0,0)$, since $y^{\frac{1}{5}}$ is continuous, **but** $f_y(t,y)$ blows up! So the theorem is inapplicable and we are **not** guaranteed a unique solution.

(Sometimes, but not always, we have solutions (but no thanks to the theorem), and they are not guaranteed to be unique!)