Dr. Z.'s Calc4 Lecture 3 Handout: Separable Differential Equations

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A **general** first-order diff.eq. has the form

$$\frac{dy}{dx} = f(x,y) \quad ,$$

where $f(x,y)$ is a function of **both** x and y, and x and y are intertwined. If you are extremely lucky, then $f(x,y)$ **only** depends on y, not on x, and then the diff.eq. is easy to solve (these are called *autonomous* equations). If you are not quite as lucky, but still pretty lucky, then $f(x,y)$ can be written as **product** $A(x)B(y)$ (or **quotient** $A(x)/B(y)$) where $A(x)$ **only** depends on x and $B(y)$ **only** depends on y.

Such diff.eqs. are called **separable**, and the method for solving this special class of diff.eqs. is called **separation of variables**.

To solve such a separable diff.eq.

$$y' = \frac{A(x)}{B(y)} \quad or \quad y' = A(x)B(y) \quad etc.$$

you replace y' by $\frac{dy}{dx}$. Treat dy and dx as algebraic quantities and **separate** the x part from the y part.

$$\frac{dy}{dx} = \frac{A(x)}{B(y)} \quad or \quad \frac{dy}{dx} = A(x)B(y) \quad etc.$$

implies

$$B(y)dy = A(x)dx \quad respectively$$

$$\frac{dy}{B(y)} = A(x)dx \quad etc.$$

Then you apply the integral sign to both sides, do the integration (if possible) and then do the algebra (if possible).

Problem 3.1: Solve the differential equation

$$y' = y^2 \sec x$$

Step 1. In this problem:

$$\frac{dy}{dx} = y^2 \sec x$$

implies

$$\frac{dy}{y^2} = \sec x \, dx$$

which is the same as

$$y^{-2} \, dy = \sec x \, dx$$

Step 2. Apply the Integral sign to both sides, and perform the integration. Only put the $+C$ on one side.

$$\int y^{-2}\,dy = \int \sec x\,dx \quad ,$$

gives

$$\frac{-1}{y} = \ln|\sec x + \tan x| + C$$

Step 3 If possible, solve for y. Otherwise leave it in implicit form. If there is an initial condition then plug it in and solve for C. If nothing is mentioned (like in this problem), then leave C alone. So:

$$y = \frac{-1}{\ln|\sec x + \tan x| + C}$$

Ans. to 3.1 $y = \frac{-1}{\ln|\sec x+\tan x|+C}$.

A typical application is to find an equation of the curve that passes through the point (a, b) and whose slope at (x, y) is given by an expression of the form $A(y)/B(x)$.

Problem 3.2: Find an equation of the curve that passes through the point $(1, 1)$ and whose slope at (x, y) is y^2/x^3.

Step 1. Slope is derivative, so set it equal to $\frac{dy}{dx}$. Treat dy and dx as algebraic quantities and **separate** the x part from the y part.

$$\frac{dy}{dx} = \frac{y^2}{x^3}$$

implies

$$\frac{dy}{y^2} = \frac{dx}{x^3}$$

which is the same as

$$y^{-2}\,dy = x^{-3}\,dx$$

step 2 Apply the Integral sign to both sides, and perform the integration. Only put the $+C$ on one side.

$$\frac{y^{-1}}{-1} = \frac{x^{-2}}{-2} + C$$

which gives

$$\frac{-1}{y} = \frac{-1}{2x^2} + C$$

Step 3. Plug in the point $(x = a, y = b)$ and solve for C. Plug back that value for C and try to express y in terms of x if possible. Otherwise leave it in implicit form.

$$\frac{-1}{1} = \frac{-1}{2 \cdot 1^2} + C$$

giving $C = -1/2$. Incorporating that C gives

$$\frac{-1}{y} = \frac{-1}{2x^2} - \frac{1}{2}$$

and algebra gives

$$y = \frac{2x^2}{1+x^2} \quad .$$

Ans. to 3.2 $\frac{2x^2}{1+x^2}$.