Dr. Z.'s Calc4 Lecture 21 Handout:
The Case of Repeated Roots When Solving Homogeneous Linear Systems with Constant Coefficients

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It sometimes happens that when we try to find solutions of homog. systems of linear diff.eqs. with constant coefficients, of the form

$$\mathbf{x}(t) = \mathbf{v}\, e^{rt} \quad ,$$

where $\mathbf{v}$ is a **CONSTANT** vector and r is some number, and follow the procedure of Lecture 19, trying to find the eignevalues of the matrix $\mathbf{P}$, the characteristic equation has **repeated roots**.

For each such repeated eigenvalue r, we can get at least one eignevector, $\mathbf{v}$ and we immediately get **one** fundamental solution. If it is repeated k times, and we get a basis of k independent eigenvectors, then it is exactly as before. But if there are fewer we have to proceed as follows.

We will only consider double roots in this class. If r is a double eigenvalue with **one**-dimensional eigenspace, i.e., up to constant multiples, we have **one** eigenvector, let's call it $\mathbf{v}$, then **one** fundamental soltion is

$$\mathbf{x}_1(t) = \mathbf{v}\, e^{rt} \quad .$$

To find another one, $\mathbf{x}_2(t)$, we try out the **template**

$$\mathbf{x}_2(t) = \mathbf{v}\, te^{rt} + \mathbf{u}\, e^{rt} \quad ,$$

for some **yet to be determined** vector $\mathbf{u}$.

We have

$$\mathbf{x}_2'(t) = \mathbf{v}\, rte^{rt} + \mathbf{v}\, e^{rt} + r\mathbf{u}\, e^{rt} \quad ,$$

Since we want

$$\mathbf{x}_2'(t) = \mathbf{P}\mathbf{x}_2(t) \quad ,$$

we have

$$\mathbf{v}rte^{rt} + \mathbf{v}e^{rt} + r\mathbf{u}e^{rt} \;=\; \mathbf{P}(\mathbf{v}te^{rt} + \mathbf{u}e^{rt})$$

But $r\mathbf{v} = \mathbf{P}\mathbf{v}$, so

$$\mathbf{v}e^{rt} \;=\; (\mathbf{P} - \mathbf{rI})\, \mathbf{u}e^{rt} \quad .$$

Dividing by e^{rt} we get a vector equation for the vector $\mathbf{u}$:

$$(\mathbf{P} - r\mathbf{I})\, \mathbf{u} = \mathbf{v} \quad .$$

Problem 21.1

Find the general solution of the system

$$\mathbf{x}'(t) = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}(t) \quad .$$

Step 1. Write down the matrix of coefficients, and set-up the **characteristic equation**.

$$\mathbf{P} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$$

$$\det \begin{pmatrix} 3-r & -4 \\ 1 & -1-r \end{pmatrix} = 0 \quad .$$

Step 2. Compute the determinant, and solve the characteristic equation, finding the eigenvalues.

$$(3-r)(-1-r) - (-4)(1) = 0 \quad ,$$

$$(r-3)(r+1) + 4 = 0$$

$$r^2 - 2r - 3 + 4 = 0 \quad ,$$

$$r^2 - 2r + 1 = 0 \quad .$$

$$(r-1)^2 = 0 \quad .$$

So we have a **repeated** eigenvalue $r = 1$.

Step 3. Find the eigenvector (or eigenvectors) corresponding to that root.

$$\begin{pmatrix} 3-r & -4 \\ 1 & -1-r \end{pmatrix} \quad .$$

becomes

$$\begin{pmatrix} 3-1 & -4 \\ 1 & -1-1 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \quad .$$

We have to find a vector $(a_1, a_2)^T$ such that

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad .$$

Spelling it out:

$$2a_1 - 4a_2 = 0 \quad , \quad a_1 - 2a_2 = 0 \quad .$$

These two equations are mulitple of each other, so it is enough to consider one of them, so let's pick the second, that is simpler. We have $a_1 = 2a_2$. So taking $a_2 = 1$ we get that $a_1 = 2$.

So an eigenvector corresponding to $r = 1$ is $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

So **one** fundamental solution is

$$\mathbf{x}_1(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t$$

Since we have only **one** (linearly independent) eigenvector, we must find **u**.

Step 4. Putting $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ we have to solve

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad .$$

Spelling it out, we have to solve

$$2u_1 - 4u_2 = 2 \quad , \quad u_1 - 2u_2 = 1 \quad .$$

Note that the second equation is a multiple of the first, and there are infinitely many answers, so let's put $u_2 = 0$, $u_1 = 1$. and we get $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(**Note**: We could have taken anything for u_2, it is just simplest to take $u_2 = 0$.)

So a second fundamental solution is

$$\mathbf{x}_2(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t \quad .$$

Finally, the **general solution** is $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ where c_1, c_2 are arbitrary constants, so the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2(\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t)$$

Ans. to 21.1: $\mathbf{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2(\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t) \quad .$

Problem 21.2

Solve the initial value system

$$\mathbf{x}'(t) = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}(t) \quad , \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad .$$

Steps 1-4. Find the general solution exactly as in Problem 21.1.

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2(\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t)$$

Step 5: Plug in $t = 0$ (or, in general $t = t_0$, if the initial condition is no at 0).

$$\mathbf{x}(0) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^0 + c_2(\begin{pmatrix} 2 \\ 1 \end{pmatrix} 0 \cdot e^0 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^0) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Step 6: Set it equal to the vector $\mathbf{x}(0)$ given by the problem,

$$\begin{pmatrix} 2c_1 + c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad ,$$

and spelled-out:

$$2c_1 + c_2 = 1 \quad , \quad c_1 = 1 \quad ,$$

and solve for c_1, c_2.

Here $c_1 = 1$ and $c_2 = -1$.

Step 7. Go back to the general solution and enter the c_1, c_2 that you just found.

$$\mathbf{x}(t) = \mathbf{x}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t - (\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t)$$

$$= e^t \begin{pmatrix} 1 - 2t \\ 1 - t \end{pmatrix}$$

Ans. to Problem 21.2 : $\mathbf{x}(t) = e^t \begin{pmatrix} 1 - 2t \\ 1 - t \end{pmatrix}$ or

$x_1(t) = (-2t + 1)e^t \quad , \quad x_2(t) = (-t + 1)e^t.$