Dr. Z.'s Calc4 Lecture 20 Handout:
The Case of Complex Roots When Solving Homogeneous Linear Systems
with Constant Coefficients

By Doron Zeilberger

It often happens that when we try to find solutions of homog. systems of linear diff.eqs. with constant coefficients, of the form

$$\mathbf{x}(t) = \mathbf{v}\, e^{rt} \quad ,$$

where $\mathbf{v}$ is a **CONSTANT** vector and r is some number, and follow the procedure of Lecture 19, trying to find the eignevalues of the matrix $\mathbf{P}$, the characteristic equation has **complex roots**. Since the matrix $\mathbf{P}$ has real entries, the roots come in complex-conjuate pairs $\lambda \pm i\mu$. The good news is that we only need to consider **one** eigenvalue of each of these pairs (so for systems of 2 equations and 2 unknown functions, just one). The bad news is that we need to use **complex calculations**, always keeping in mind that $i^2 = -1$.

Problem 20.1

Find the general solution of the system

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}(t) \quad .$$

Step 1. Write down the matrix of coefficients, and set-up the **characteristic equation**.

$$\mathbf{P} = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}$$

$$\det \begin{pmatrix} 1-r & -5 \\ 1 & -3-r \end{pmatrix} = 0 \quad .$$

Step 2. Compute the determinant, and solve the characteristic equation, finding the eigenvalues.

$$(1-r)(-3-r) - (-5)(1) = 0 \quad ,$$

$$(r-1)(r+3) + 5 = 0$$

$$r^2 + 2r - 3 + 5 = 0 \quad ,$$

$$r^2 + 2r + 2 = 0 \quad .$$

Using the famous formula for finding the roots of a quadratic equation

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we get

$$r_1, r_2 = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2 \cdot 1} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i \quad .$$

Step 3. We only need to find the eignevector corresponding to $r_1 = -1+i$. When $r = -1+i$,

$$\begin{pmatrix} 1-r & -5 \\ 1 & -3-r \end{pmatrix} \quad .$$

becomes

$$\begin{pmatrix} 1-(-1+i) & -5 \\ 1 & -3-(-1+i) \end{pmatrix} = \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \quad .$$

We have to find a vector $(a_1, a_2)^T$ such that

$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad .$$

Spelling it out:

$$(2-i)a_1 - 5a_2 = 0 \quad , \quad a_1 - (2+i)a_2 = 0 \quad .$$

These two equations are mulitple of each other, so it is enough to consider one of them, so let's pick the second, that is simpler. We have $a_1 = (2+i)a_2$. So taking $a_2 = 1$ we get that $a_1 = 2+i$.

So an eigenvector corresponding to $r_1 = -1+i$ is $\mathbf{v}_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$.

Step 4. The (one, specific) solution that we found so far is

$$\mathbf{x}(t) = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{(-1+i)t}$$

Now we write

$$e^{(-1+i)t} = e^{-t}e^{it},$$

and use Euler's famous formula

$$e^{it} = \cos t + i \sin t \quad ,$$

to get

$$\mathbf{x}(t) = e^{-t} \begin{pmatrix} 2+i \\ 1 \end{pmatrix} (\cos t + i \sin t)$$

This equals

$$\mathbf{x}(t) = e^{-t} \begin{pmatrix} (2+i)(\cos t + i \sin t) \\ (\cos t + i \sin t) \end{pmatrix}$$

Doing the complex algebra, this is

$$\mathbf{x}(t) = e^{-t} \begin{pmatrix} 2\cos t + 2i(\sin t) + i \cos t - \sin t \\ \cos t + i \sin t \end{pmatrix} = e^{-t} \begin{pmatrix} (2\cos t - \sin t) + i(2\sin t + \cos t) \\ \cos t + i \sin t \end{pmatrix}$$

We now separate the **real** and **imaginary** parts

$$\mathbf{x}(t) = e^{-t} \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + ie^{-t} \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix} \quad .$$

Obviously the real and imaginary parts are **linearly independent**, so we found two independent solutions

$$\mathbf{x}_1(t) = e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} \quad ,$$

$$\mathbf{x}_2(t) = e^{-t}\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix} \quad .$$

The **general solutions** is simply $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$, where c_1, c_2 are **arbitrary constants**.

In this problem it is

$$\mathbf{x}(t) = c_1 e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix} \quad .$$

Ans. to 20.1: $\mathbf{x}(t) = c_1 e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$

In scalar notation:

$$x_1(t) = e^{-t}(c_1(2\cos t - \sin t) + c_2(\cos t + 2\sin t)) \quad , \quad x_2(t) = e^{-t}(c_1\cos t + c_2\sin t) \quad .$$

Problem 20.2

Solve the initial value system

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}\mathbf{x}(t) \quad , \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad .$$

Steps 1-4. Find the general solution exactly as in Problem 20.1.

$$\mathbf{x}(t) = c_1 e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix} \quad .$$

Step 5: Plug in $t = 0$ (or, in general $t = t_0$, if the initial condition is no at 0).

$$\mathbf{x}(0) = c_1 e^{-0}\begin{pmatrix} 2\cos 0 - \sin 0 \\ \cos 0 \end{pmatrix} + c_2 e^{-0}\begin{pmatrix} 2\sin 0 + \cos 0 \\ \sin 0 \end{pmatrix} = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2c_1 + c_2 \\ c_1 \end{pmatrix} \quad .$$

Step 6: Set it equal to the vector $\mathbf{x}(0)$ given by the problem,

$$\begin{pmatrix} 2c_1 + c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad ,$$

and spelled-out:

$$2c_1 + c_2 = 1 \quad , \quad c_1 = 1 \quad ,$$

and solve for c_1, c_2.

Here $c_1 = 1$ and $c_2 = -1$.

Step 7. Go back to the general solution and enter the c_1, c_2 that you just found.

$$\mathbf{x}(t) = e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} - e^{-t}\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix} \quad .$$

$$= e^{-t}\begin{pmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{pmatrix}$$

Ans. to Problem 20.2 : $\mathbf{x}(t) = e^{-t}\begin{pmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{pmatrix}$ or

$x_1(t) = e^{-t}(\cos t - 3\sin t) \quad , \quad x_2(t) = e^{-t}(\cos t - \sin t).$