Dr. Z.'s Calc4 Lecture 2 Handout: Method of Integrating Factors for First-Order Linear Equations

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Theory: After possibly dividing the coefficient of $y'(t)$, any first-order linear diff.eq. has the format

$$y'(t) + p(t)y(t) = g(t) \quad ,$$

for some functions $p(t)$ and $g(t)$ of t. The **Integrating Factor**, $I(t)$, is

$$I(t) = e^{\int p(t)\,dt} \quad .$$

What's nice about it is that $I'(t) = p(t)I(t)$ (by the chain rule, and the fundamental theorem of Calculus). Multiplying both sides of the diff.eq. by $I(t)$ yields

$$I(t)y'(t) + p(t)I(t)y(t) = I(t)g(t) \quad .$$

which is the same as

$$I(t)y'(t) + I'(t)y(t) = I(t)g(t) \quad .$$

which is the same as

$$(I(t)y(t))' = I(t)g(t) \quad .$$

Integrating both sides gives

$$I(t)y(t) = \int I(t)g(t)\,dt \quad ,$$

and dividing both sides by $I(t)$ gives:

$$y(t) = \frac{\int I(t)g(t)\,dt}{I(t)} \quad .$$

Note: As you know, it is not always possible to integrate nicely, so do your best, and if you can't do it, leave it in terms of integral signs.

Problem 2.1: Find the general solution of the diff.eq.

$$ty' + 3y = t^2 \quad .$$

Step 1: If the coefficient of y' is not 1 divide the diff.eq. by the coeff. of y' getting an equivalent diff.eq. with that property.

$$y' + \frac{3}{t}y = t$$

Step 2: Having made sure that y' is all by itself, decide what is $p(t)$, the coeff. of y.

$$p(t) = \frac{3}{t} \quad .$$

Step 3: Integrate $p(t)$, using your calc2 skills. Don't bother about the $+C$ (arbitrary constant), since all we need is **one** magic integrating factor, not all of them!

$$\int p(t)\,dt = \int \frac{3}{t}\,dt = 3\ln t \quad .$$

Step 4: Exponentiate $\int p(t)\,dt$, to get $I(t)$. Remember that $exp(\ln Whatever) = Whatever$.

$$I(t) = exp(\int p(t)\,dt) = exp(3\ln t) = exp(\ln t^3) = t^3 \quad .$$

Step 5: (first way). Plug into the general formula

$$y(t) = \frac{\int I(t)g(t)\,dt}{I(t)} \quad .$$

$g(t)$ is the function on the right, so in this problem $g(t) = t$. So

$$y(t) = \frac{\int I(t)g(t)\,dt}{I(t)} = \frac{\int (t^3)t\,dt}{t^3} = \frac{\int t^4\,dt}{t^3} = \frac{\frac{1}{5}t^5 + C}{t^3} = \frac{1}{5}t^2 + \frac{C}{t^3} \quad .$$

Step 5: (second way). Multiply the (reduced) diff.eq. by $I(t)$:

$$t^3y' + t^3\frac{3}{t}y = t^4 \quad ,$$

simplifying

$$t^3y' + 3t^2y = t^4 \quad ,$$

and use the product rule:

$$(t^3y)' = t^4 \quad .$$

Integrate both sides:

$$t^3y = \int t^4\,dt = \frac{1}{5}t^5 + C \quad ,$$

and finally divide by $I(t)$:

$$y(t) = \frac{1}{5}t^2 + \frac{C}{t^3} \quad .$$

Ans. to 2.1: $y(t) = \frac{1}{5}t^2 + \frac{C}{t^3}$, where C is an **arbitrary constant**.

Problem 2.2: Find a solution solution to the initial value diff.eq.

$$ty' + 3y = t^2 \quad , \quad y(1) = 1 \quad .$$

Solution of 2.2: The beginning is exactly the same as before. We find the general solution featuring a general (arbitrary) constant C. Now you incorporate the extra information given by the initial conditions. When $t = 1$, $y = 1$ so

$$y(1) = \frac{1}{5} \cdot 1^2 + \frac{C}{1^3} = 1$$

So

$$\frac{1}{5} + C = 1 \quad .$$

Solving for C gives $C = 1 - \frac{1}{5} = \frac{4}{5}$. Having found the actual value of C, you go back to the general solution, and replace C by $\frac{4}{5}$.

Ans. to 2.2: $y(t) = \frac{1}{5}t^2 + \frac{4}{5t^3}$.