Dr. Z.'s Calc4 Lecture 19 Handout: Homogeneous Linear Systems with Constant Coefficients

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The general format of a **linear homogeneous** system is

$$\mathbf{x}'(t) = \mathbf{P}(t)\,\mathbf{x}(t) \quad ,$$

where $\mathbf{P}(t)$ is a matrix of FUNCTIONS of t. If we are lucky and all the entries of $\mathbf{P}(t)$ are **constant** functions, we have a **constant-coefficients** system, whose format is

$$\mathbf{x}'(t) = \mathbf{P}\mathbf{x}(t) \quad ,$$

where $\mathbf{P}$ is a *matrix of NUMBERS*.

Let's try to find *some* solution of such a system, and inspired by single equations, let's try an **exponential** solution

$$\mathbf{x}(t) = \mathbf{v}e^{rt} \quad ,$$

where $\mathbf{v}$ is a **CONSTANT** vector and r is some number. Plugging this template into $\mathbf{x}'(t) = \mathbf{P}\mathbf{x}(t)$, we get

$$r\mathbf{v}e^{rt} = \mathbf{P}\mathbf{v}e^{rt} \quad .$$

Dividing both sides by e^{rt} we get

$$r\mathbf{v} = \mathbf{P}\mathbf{v} \quad .$$

but this means that r is an **eigenvalue** of our matrix $\mathbf{P}$ and $\mathbf{v}$ is a corresponding eigenvector.

So by finding all the eigenvalues $r_1, r_2, \dots, r_n$ each with their corresponding eigenvectors $\mathbf{v}_1, \dots \mathbf{v}_n$, we get n linearly independent solutions (if all the eigenvalues are distinct), and we can write down the **general solution**.

$$\mathbf{x}(t) = c_1\mathbf{v_1}e^{r_1 t} + \dots + c_n\mathbf{v_n}e^{r_n t} \quad .$$

If there are also *initial conditions*, then we have to use linear algebra to find the numbers $c_1, \dots, c_n$.

In problems that you are likely to get the system has either $n = 2$ or $n = 3$.

Problem 19.1

Find the general solution of the system

$$\mathbf{x}'(t) = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}(t) \quad .$$

Step 1. Write down the matrix of coefficients, and set-up the **characteristic equation**.

$$\mathbf{P} = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$$

$$\det \begin{pmatrix} 5-r & -1 \\ 3 & 1-r \end{pmatrix} = 0 \quad .$$

Step 2. Compute the determinant, and solve the characteristic equation, finding the eigenvalues.

$$(5-r)(1-r) - (-1)(3) = 0 \quad ,$$

$$5 - 6r + r^2 + 3 = 0 \quad ,$$

$$r^2 - 6r + 8 = 0 \quad ,$$

$$(r-2)(r-4) = 0 \quad ,$$

so the eigenvalues are $r = 2$ and $r = 4$.

Step 3. For each of the eigenvalues, find a corresponding eigenvector.

When $r = 2$, the matrix

$$\begin{pmatrix} 5-r & -1 \\ 3 & 1-r \end{pmatrix}$$

becomes

$$\begin{pmatrix} 5-2 & -1 \\ 3 & 1-2 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}$$

We have to find a vector $(a_1, a_2)^T$ such that

$$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad .$$

Spelling it out:

$$3a_1 - a_2 = 0 \quad , \quad 3a_1 - a_2 = 0 \quad .$$

(note the repeat), we get $a_2 = 3a_1$, so the vector is $\begin{pmatrix} a_1 \\ 3a_1 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Taking $a_1 = 1$ (we can take any **non-zero** number, but 1 is usually (but not always) the simplest). So we have

an eigenvector corresponding to $r_1 = 2$ is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

When $r = 4$, the matrix

$$\begin{pmatrix} 5-r & -1 \\ 3 & 1-r \end{pmatrix}$$

becomes

$$\begin{pmatrix} 5-4 & -1 \\ 3 & 1-4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}$$

We have to find a vector $(a_1, a_2)^T$ such that

$$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad .$$

Spelling it out:

$$a_1 - a_2 = 0 \quad , \quad 3a_1 - 3a_2 = 0 \quad .$$

(note that the second equation is a multiple of the first), we get $a_2 = a_1$, so the vector is $\begin{pmatrix} a_1 \\ a_1 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Taking $a_1 = 1$ (we can take any **non-zero** number, but 1 is usually (but not always) the simplest). So we have

an eigenvector corresponding to $r_2 = 4$ is $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Step 4. Write down the general solution

$$\mathbf{x}(t) = c_1 \mathbf{v_1} e^{r_1 t} + c_2 \mathbf{v_2} e^{r_2 t} \quad ,$$

where c_1, c_2 are **arbitrary constants**.

In this problem we have:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} \quad .$$

Ans. to 19.1: $\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$, and in spelled-out notation

$$x_1(t) = c_1 e^{2t} + c_2 e^{4t} \quad , \quad x_2(t) = 3c_1 e^{2t} + c_2 e^{4t} \quad .$$

Problem 19.2

Solve the initial value system

$$\mathbf{x}'(t) = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}(t) \quad , \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad .$$

Steps 1-4. Find the general solution exactly as in Problem 19.1.

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} \quad .$$

Step 5: Plug in $t = 0$ (or, in general $t = t_0$, if the initial condition is no at 0).

$$\mathbf{x}(0) = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2 \cdot 0} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4 \cdot 0} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad .$$

Step 6: Set it equal to the vector $\mathbf{x}(0)$ given by the problem,

$$c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad .$$

and spell-it out:

$$\begin{pmatrix} c_1 + c_2 \\ 3c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$c_1 + c_2 = 2 \quad , \quad 3c_1 + c_2 = -1 \quad ,$$

and solve for c_1, c_2.

Subtracting the first equation from the second, we get $2c_1 = -3$, so $c_1 = -\frac{3}{2}$, and using the first equation $c_2 = 2 - c_1 = \frac{7}{2}$.

Step 7. Go back to the general solution and enter the c_1, c_2 that you just found.

$$\mathbf{x}(t) = -\frac{3}{2} \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + \frac{7}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} \quad .$$

This is an acceptable answer. If you are asked to spell it out in scalar form, you go back to the scalar-form (non-vector) general solution

$$x_1(t) = c_1 e^{2t} + c_2 e^{4t} \quad , \quad x_2(t) = 3c_1 e^{2t} + c_2 e^{4t} \quad ,$$

and plug-in the c_1 and c_2 that you found.

$$x_1(t) = -\frac{3}{2} e^{2t} + \frac{7}{2} e^{4t} \quad , \quad x_2(t) = -\frac{9}{2} e^{2t} + \frac{7}{2} e^{4t} \quad ,$$

Ans. to Problem 19.2 : $\mathbf{x}(t) = -\frac{3}{2} \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + \frac{7}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$ (in vector notation),

or

$x_1(t) = -\frac{3}{2} e^{2t} + \frac{7}{2} e^{4t}$, $x_2(t) = -\frac{9}{2} e^{2t} + \frac{7}{2} e^{4t}$ (in scalar notation).