Dr. Z.'s Calc4 Lecture 18 Handout: Systems of First-Order Linear Equations

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The general **format** of a *system* of n **linear** differential equations with n unknown functions $x_1(t), \ldots, x_n(t)$ is:

$$x'_1(t) = p_{11}(t)x_1(t) + \ldots + p_{1n}(t)x_n(t) + g_1(t) \quad ,$$

$$\ldots$$

$$\ldots$$

$$x'_n(t) = p_{11}(t)x_1(t) + \ldots + p_{1n}(t)x_n(t) + g_n(t) \quad ,$$

For some (often constant, but in general not) n^2 functions $p_{11}(t), \ldots, p_{nn}(t)$ and n functions $g_1(t), \ldots, g_n(t)$.

Using **matrix shortand notation** we "compactify" the n unknwon functions $x_1(t), \ldots, x_n(t)$ into **ONE** vector of functions, and call it $\mathbf{x}(t)$.

$$\mathbf{x}(t) = [x_1(t), \ldots, x_n(t)]^T$$

and we condense all the n^2 functions $p_{11}(t), \ldots, p_{nn}(t)$ into **ONE** matrix

$$\mathbf{P} = \begin{pmatrix} p_{11}(t) & \ldots & p_{1n}(t) \\ \ldots & \ldots & \ldots \\ p_{n1}(t) & \ldots & p_{nn}(t) \end{pmatrix}$$

and we condense the n functions $g_1(t), \ldots, g_n(t)$ into **ONE** vector.

$$\mathbf{g}(t) = [g_1(t), \ldots, g_n(t)]^T$$

Then our system can be written, very compactly as **ONE** equation (but in matrix notation)

$$\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t) + \mathbf{g}(t) \quad .$$

If all the $g_i(t)$ are zero, in other words, if the vector $\mathbf{g}(t)$ is the zero vector, the system is **homogeneous**.

Problem 18.1: Write the follwing system in matrix notation

$$x'_1(t) = -t^2 x_3(t) + 2t\, x_2(t) + t^2$$

$$x'_2(t) = -x_3(t) - x_1(t) + \cos t$$

$$x'_3(t) = -x_2(t) + \tan t\, x_1(t) + \sin t$$

in matrix notation.

Step 1. Rearrange the terms so that there are in order and put 0 if one of the functions is missing

$$x_1'(t) = 0\,x_1(t) + 2t x_2(t) - t^2\,x_3(t) + t^2 \quad ,$$

$$x_2'(t) = -x_1(t) + 0\,x_2(t) - \cos t\,x_3(t) + 0 \quad ,$$

$$x_3'(t) = \tan t\,x_1(t) - x_2(t) + 0\,x_3(t) + \sin t \quad .$$

Step 2: Figure out the coefficient matrix $\mathbf{P}$ and write it in matrix notation

$$\mathbf{x}'(t) = \begin{pmatrix} 0 & 2t & -t^2 \\ -1 & 0 & -\cos t \\ \tan t & -1 & 0 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} t^2 \\ 0 \\ \sin t \end{pmatrix} \quad .$$

Important Facts about Homogeneous Systems

1. $\mathbf{x}(t)$, is a solution of the homogeneous matrix diff.eq. $\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t)$, then so is any scalar multiple $c\mathbf{x}(t)$

2. $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ are solutions of the homogeneous matrix diff.eq. $\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t)$, then so is there sum $\mathbf{x}_1(t) + \mathbf{x}_2(t)$

Repeating this, it turns out that we have any n specific (vector) solutions

$$\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$$

then for *any* constants $c_1, c_2, \ldots, c_n(t)$, the **linear combination**

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \ldots + c_n\mathbf{x}_n(t)$$

is yet another solution.

Another Important Fact

If

$$\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$$

are **linearly independent** solutions of the system, then **EVERY** (i.e. the **GENERAL**) solution, can be writen as

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \ldots + c_n\mathbf{x}_n(t) \quad .$$

To find the actual numbers $c_1, \ldots, c_n$, you need the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ for some numerical vector $\mathbf{x}_0$.