Dr. Z.'s Calc4 Lecture 16 Handout:
NonHomogeneous (Linear) Higher-Order Diff. Eqs.; Undetermined Coefficients

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In order to find the **general solution** of a (general) linear n^{th}-order diff.eq.

$$y^{(n)}(t) + p_1(t)\, y^{(n-1)}(t) + \ldots + p_n(t)\, y(t) = g(t) \quad ,$$

you find the general solution of the **corresponding** *homogeneous* version (obtained by replacing $g(t)$ by 0)

$$y^{(n)}(t) + p_1(t)\, y^{(n-1)}(t) + \ldots + p_n(t)\, y(t) = 0 \quad ,$$

that has the format $c_1 y_1(t) + c_2 y_2(t) + \ldots y_n(t)$ for some specific solutions $y_1(t), y_2(t), \ldots, y_n(t)$ (whose Wronskian is not identically zero).

Then by **hook or crook** you find **ONE** specific solution P(t) (called the **particular solution**) of the original (inhomogeneous diff.eq.). Then

$$General \quad Solution = c_1 y_1(t) + c_2 y_2(t) + \ldots + c_n y_n(t) + P(t) \quad .$$

In other words

General Solution of InHomog. Diff.Eq.= General Solution of Homog. Diff.Eq. + Particular Solution .

How to find Particular Solutions of Constant Coefficient Inhomog. linear diff.eqs.

The rules are exactly the same as for second-order equation, as described in Lecture 12. Let's recall them.

If the right side is a (specific) polynomial of degree k, your **first try** is a generic polynomial (with undetermined coefficients) of degree k. In particular if it is a constant, you try out $y(t) = A_0$.

If the right side is a (specific) exponential, $e^{\alpha t}$, your **first try** is a $A_0 e^{\alpha t}$.

If the right side is a (specific) exponential times trig, $e^{\alpha t} \sin \beta t$, or $e^{\alpha t} \cos \beta t$, or, more generally, $e^{\alpha t}(A \sin \beta t + B \cos \beta t)$ your **first try** is a $e^{\alpha t}(A_0 \sin \beta t + B_0 \cos \beta t)$.

If the right side is a (specific) exponential, times a specific polynomial of degree k, $e^{\alpha t} p(t)$, your **first try** is a $e^{\alpha t}(A_0 t^k + A_1 t^{k-1} + \ldots + A_k t^0)$.

If the **first try** does not work, multiply the proposed Particular Solution by t. If it still does not work, try and try again!

Problem 16.1: Find a particular solution of the diff.eq.

$$y^{(10)}(t) + 3\, y^{(5)}(t) + 6\, y(t) = 12$$

Solution of 16.1: Since the right side is a **constant**, you try out

$$y(t) = A_0 \quad ,$$

where A_0 is a constant yet **to be determined**.

Now $y'(t) = 0, y''(t) = 0, \ldots, y^{(10)}(t) = 0$ so, plugging into the diff.eq.

$$0 + 0 + 6 \cdot A_0 = 12$$

Solving the equation $6A_0 = 12$ we get $A_0 = 2$. So a particular solution is $y(t) = 2$.

Ans. to 16.1: a particular solution is $y(t) = 2$.

Problem 16.2: Find a particular solution of the diff.eq.

$$y^{(4)}(t) + y(t) = t^4 + t^2 + 24$$

Solution of 16.2: Since the right side is a polynomial in t of **degree 4**, you try out

$$y(t) = A_0t^4 + A_1t^3 + A_2t^2 + A_3t + A_4 \quad ,$$

where A_0, A_1, A_2, A_3, A_4 are constants yet **to be determined**.

Now

$$y(t) = A_0t^4 + A_1t^3 + A_2t^2 + A_3t + A_4 \quad ,$$

$$y'(t) = 4A_0t^3 + 3A_1t^2 + 2A_2t + A_3 \quad ,$$

$$y''(t) = 12A_0t^2 + 6A_1t + 2A_2 \quad ,$$

$$y'''(t) = 24A_0t + 6A_1 \quad ,$$

$$y^{(4)}(t) = 24A_0 \quad .$$

Substituing in the diff.eq.

$$y^{(4)}(t)+y(t) = 24A_0+(A_0t^4+A_1t^3+A_2t^2+A_3t+A_4) = A_0t^4+A_1t^3+A_2t^2+A_3t+(24A_0+A_4) \; = \; t^4+t^2+24$$

Comparing coefficinets

$$A_0 = 1 \quad , \quad A_1 = 0 \quad , \quad A_2 = 1 \quad , \quad A_3 = 0 \quad , \quad 24A_0 + A_4 = 24 \quad .$$

Solving these equation, we get $A_0 = 1, A_1 = 0, A_2 = 1, A_3 = 0, A_4 = 0$. Going back to the general template $A_0t^4 + A_1t^3 + A_2t^2 + A_3t + A_4$ we get that a particular solution is $y(t) = t^4 + t^2$.

Ans. to 16.2: a particular solution is $y(t) = t^4 + t^2$.

Problem 16.3: Find a particular solution of the diff.eq.

$$y^{(4)}(t) + 3\,y''(t) + 2\,y(t) = 60e^{2t}$$

Solution of 16.3: Since the right side is a pure **exponential function** e^{2t}, we simply try

$$y(t) = A_0 e^{2t}$$

where A_0 is a constant yet **to be determined**.

Now $y'(t) = 2A_0e^{2t}, y''(t) = 4A_0e^{2t}$, $y'''(t) = 8A_0e^{2t}, y^{(4)}(t) = 16A_0e^{2t}$ so, plugging into the diff.eq.

$$y^{(4)}(t) + 3\,y''(t) + 2\,y(t) = 16A_0e^{2t} + 3(4A_0e^{2t}) + 2A_0e^{2t} =$$

Simplifying:

$$(16A_0 + 12A_0 + 2A_0)e^{2t} = 60e^{2t}$$

$$30A_0e^{2t} = 60e^{2t} \quad .$$

Dividing by e^{2t}, $30A_0 = 60$, and solving for A_0, we get $A_0 = 2$.

Ans. to 16.3: a particular solution is $y(t) = 2e^{2t}$.

Problem 16.4: Find a particular solution of the diff.eq.

$$y'''(t) + y'(t) + y(t) = \sin 2t \quad .$$

Solution of 16.4: We try the template

$$y(t) = A\sin 2t + B\cos 2t \quad .$$

We now find the derivatives

$$y'(t) = 2A\cos 2t - 2B\sin 2t \quad .$$

$$y''(t) = -4A\sin 2t - 4B\cos 2t \quad .$$

$$y'''(t) = -8A\cos 2t + 8B\sin 2t \quad .$$

Putting this in the diff.eq.

$$y'''(t) + y'(t) + y(t) = -8A\cos 2t + 8B\sin 2t + 2A\cos 2t - 2B\sin 2t + A\sin 2t + B\cos 2t = \sin 2t \quad .$$

Collecting terms

$$(A + 6B)\sin 2t + (B - 6A)\cos 2t = \sin 2t \quad .$$

Comparing coefficients of $\sin 2t$ and $\cos 2t$ we get

$$A + 6B = 1 \quad , \quad B - 6A = 0 \quad .$$

From the second equation we get $B = 6A$. Putting in the first: $A + 6(6A) = 1$, so $37A = 1$, so $A = \frac{1}{37}$ and so $B = \frac{6}{37}$. Going back to the template, we have:

$$y(t) = \frac{1}{37}\sin 2t + \frac{6}{37}\cos 2t \quad .$$

Ans. to 16.4: a particular solution is $y(t) = \frac{1}{37}\sin 2t + \frac{6}{37}\cos 2t$.

Problem 16.5: Find a particular solution of the diff.eq.

$$y^{(6)}(t) - y(t) = e^t$$

Solution of 16.5: Since the right side is a pure **exponential function** e^t, the **first try** is

$$y(t) = A_0 e^t \quad .$$

Now

$$y'(t) = A_0 e^t \quad , \quad y''(t) = A_0 e^t \quad , y'''(t) = A_0 e^t \quad , \quad y^{(4)}(t) = A_0 e^t \quad ,$$

$$y^{(5)}(t) = A_0 e^t \quad , \quad y^{(6)}(t) = A_0 e^t \quad .$$

$$A_0 e^t - A_0 e^t = e^t \quad .$$

Simplifying, we get:

$$0 = e^{-t} \quad .$$

NONSENSE!. So the first try failed. We now have a **second try**, by multiplying the first try by t.

$$y(t) = A_0 t e^t \quad .$$

Now

$$y'(t) = A_0(t+1)e^t \quad , \quad y''(t) = A_0(t+2)e^t \quad .$$

$$y'''(t) = A_0(t+3)e^t \quad , \quad y^{(4)}(t) = A_0(t+4)e^t \quad .$$

$$y^{(5)}(t) = A_0(t+5)e^t \quad , \quad y^{(6)}(t) = A_0(t+6)e^t \quad .$$

Plugging into the diff.eq.

$$y^{(6)}(t) - y(t) = A_0(t+6)e^t - A_0 t e^t = 6A_0 e^t = e^t$$

Dividing by e^t, we get

$$6A_0 = 1 \quad ,$$

whose solution is $A_0 = \frac{1}{6}$.

Going back to the template, we get that a particular solution is $y(t) = \frac{1}{6}te^t$.

Ans. to 16.5: a particular solution is $y(t) = \frac{1}{6}te^t$.

Problem 16.6: Set-up but **do not solve** a template for a PARTICULAR solution of

$$y^{(5)}(t) - y'(t) = t^2 + 4 + e^t + \cos t + e^{3t} \quad .$$

Solution of 16.6: Since the right side consists of a polynomial of degree 2, a pure exponential, and a pure trig. The first temptation would be to try

$$y(t) = A_0 t^2 + A_1 t + A_2 + A_3 e^t + A_4 \cos t + A_5 \sin t + A_6 e^{3t} \quad ,$$

but **don't jump to conclusions!**.

The characteristic equation of the homog. version is

$$r^5 - r = 0 \quad .$$

Factoring

$$r(r-1)(r+1)(r^2+1) = 0 \quad .$$

So a fundamental set of solutions would be $1, e^t, e^{-t}, \cos t, \sin t$. Hence the A_2 , A_3, and A_4 and A_5 would disapper. All we have to do is multiply by t those terms that overlap with the general solution. So the correct template is:

$$y(t) = t(A_0 t^2 + A_1 t + A_2) + A_3 t e^t + t(A_4 \cos t + A_5 \sin t) + A_6 e^{3t} \quad .$$

Ans. to 16.6: an appropriate template for a particular solution is $y(t) = t(A_0 t^2 + A_1 t + A_2) + A_3 t e^t + t(A_4 \cos t + A_5 \sin t) + A_6 e^{3t}$.