Dr. Z.'s Calc4 Lecture 15 Handout: Homogeneous Equations With Constant Coefficients

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When the coefficients are **constant functions** (or constants for short), then our homog. linear diff.eq. of order n looks as follows

$$a_0 y^{(n)}(x) + a_1 y^{(n-1)}(x) + \ldots + a_{n-1} y'(x) + a_n y(x) = 0 \quad ,$$

for some **numbers** (alias constant functions) $a_0, a_1, \ldots, a_n$.

To solve it, you form the **characteristic equation**, by replacing $y(x)$ by 1, $y'(x)$ by r, $y''(x)$ by r^2, $y'''(x)$ by r^3 etc..

Then you solve it (either by inspection, or with Maple or Matlab).

Case 1: **Real and Distinct Roots**, let's call them $r_1, r_2, \ldots, r_n$. Then the **general solution** is

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots + c_n e^{r_n x} \quad ,$$

where $c_1, c_2, \ldots, c_n$ are *arbitrary constants.*

Case 2: **One or more Root is repeated**: If the root r_1 is a *double root* then the corresponding solutions are $e^{r_1 x}, x e^{r_1 x}$. If the root r_1 is a *triple root* then the corresponding solutions are $e^{r_1 x}, x e^{r_1 x}, x^2 e^{r_1 x}$. If the root r_1 is a *quadruple root* the the corresponding solutions are $e^{r_1 x}, x e^{r_1 x}, x^2 e^{r_1 x}, x^3 e^{r_1 x}$. Etc.

Case 3: **Complex Roots:** For every pair $\lambda \pm i\mu$ correspond the solutions $e^{\lambda x} \cos \mu x$ and $e^{\lambda x} \sin \mu x$.

Problem 15.1: Find the general solution of

a. $y'''(t) - y'(t) = 0$

b. $y'''(t) - 6y''(t) + 11y'(t) - 6y(t) = 0$

Solution of 15.1a: The **characteristic equation** is $r^3 - r = 0$. Factorizing, we get $r(r^2 - 1) = 0$ and then $r(r-1)(r+1) = 0$, so the three roots are *distinct* and they are $r = -1$, $r = 0$, and $r = 1$ corresponding to the fundamental solutions are $e^{-t}, e^{0t} = 1, e^t$, and the general solution is a generic *linear combination*

$$y(t) = c_1 e^{-t} + c_2 + c_3 e^t \quad .$$

Ans. to 15.1a: $y(t) = c_1 e^{-t} + c_2 + c_3 e^t$.

Solution of 15.1b: The **characteristic equation** is $r^3 - 6r^2 + 11r - 6 = 0$. Factorizing, we get $(r-1)(r-2)(r-3) = 0$ and so the three roots are *distinct* and they are $r = 1$, $r = 2$, and $r = 3$

corresponding to the fundamenta; solutions are e^t, e^{2t}, e^{3t}, and the general solution is a generic *linear combination*

$$y(t) = c_1e^t + c_2e^{2t} + c_3e^{3t} \quad .$$

Ans. to 15.1b $y(t) = c_1e^t + c_2e^{2t} + c_3e^{3t}$

Problem 15.2: Find the general solution of

a. $y''' - 2y''(x) + y'(x) = 0$

b. $y^{(4)} - 3y^{(3)}(x) + 3y''(x) - y'(x) = 0$

Solution of 15.2a: The **characteristic equation** is $r^3 - 2r^2 + r = 0$. Factorizing, we get $r(r-1)^2 = 0$ and so the root $r = 0$ is a single-root, but the root $r = 1$ is a double-root. So the fundamental solutions are $e^{0\cdot t} = 1, e^t, te^t$, and the general solution is a generic *linear combination*

$$y(t) = c_1 + c_2e^t + c_3te^t \quad .$$

Ans. to 15.2a: $y(t) = c_1 + c_2e^t + c_3te^t$.

Solution of 15.2b: The **characteristic equation** is $r^4 - 3r^3 + 3r^2 - r = 0$. Factorizing, we get $r(r-1)^3 = 0$ and so the root $r = 0$ is a single-root, but the root $r = 1$ is a triple-root. So the solutions are $e^{0\cdot t} = 1, e^t, , te^t, t^2e^t$, and the general solution is a generic *linear combination*

$$y(t) = c_1 + c_2\, e^t + c_3\; te^t + c_4\, t^2e^t \quad .$$

Ans. to 15.2b: $y(t) = c_1 + c_2e^t + c_3te^t + c_4t^2e^t$.

Problem 15.3: Find the general solution of

a. $y^{(4)}(x) - y(x) = 0$

b. $y^{(4)} + 2y''(x) + y(x) = 0$

Solution of 15.3a: The **characteristic equation** is $r^4 - 1 = 0$. Factorizing, we get $(r^2-1)(r^2+1) = 0$, and then $(r-1)(r+1)(r^2+1) = 0$. and so the root $r = -1$, $r = 1$ and $r = \pm i = 0 \pm 1 \cdot i$.

So the solutions are $e^{-t}, e^t, e^{0\cdot t}\cos t, e^{0\cdot t}\sin t$, so they are $e^{-t}, e^t, \cos t, \sin t$, and the general solution is a generic *linear combination*

$$y(t) = c_1\, e^{-t} + c_2\, e^t + c_3\, \cos t + c_4\, \sin t \quad .$$

Ans. to 15.3a: $y(t) = c_1\, e^{-t} + c_2\, e^t + c_3\, \cos t + c_4\, \sin t$.

Solution of 15.3b: The **characteristic equation** is $r^4 + 2r^2 + 1 = 0$. Factorizing, we get $(r^2 + 1)^2 = 0$, and so the roots are: $r = \pm i = 0 \pm 1 \cdot i$, each repeated twice!

So the solutions are $\cos t, \sin t, t\cos t, t\sin t$, and the general solution is a generic *linear combination*

$$y(t) = c_1 \cos t + c_2 \sin t + c_3\, t\cos t + c_4\, t\sin t$$

Ans. to 15.3a: $y(t) = c_1 \cos t + c_2 \sin t + c_3\, t\cos t + c_4\, t\sin t$

Problem 15.1a': Solve the initial value problem

a. $y'''(t) - y'(t) = 0 \quad ; \quad y(0) = 3 \quad , \quad y'(0) = 0 \quad , \quad y''(0) = 2$

Solutions of 15.1a': We first do 15.1a getting that the general solution is

$$y(t) = c_1 e^{-t} + c_2 + c_3 e^t \quad .$$

We now find expressions for $y'(t)$ and $y''(t)$:

$$y'(t) = -c_1 e^{-t} + c_3 e^t \quad ,$$

$$y''(t) = c_1 e^{-t} + c_3 e^t \quad .$$

We now plug-in $t = 0$

$$y(0) = c_1 + c_2 + c_3 \quad ,$$

$$y'(0) = -c_1 + c_3 \quad ,$$

$$y''(0) = c_1 + c_3 \quad .$$

We now use the initial conditions, $y(0) = 3, y'(0) = 0, y''(0) = 2$ getting the system of linear equations

$$c_1 + c_2 + c_3 = 3 \quad ,$$

$$-c_1 + c_3 = 0 \quad ,$$

$$c_1 + c_3 = 2 \quad ,$$

whose solution is $c_1 = 1, c_2 = 1, c_3 = 1$. We now go back to the general solution and substitute for these.

$$y(t) = 1 \cdot e^{-t} + 1 + 1 \cdot e^t = e^{-t} + 1 + e^t \quad .$$

Ans. to 15.1a': $y(t) = e^{-t} + 1 + e^t$.