Dr. Z.'s Calc4 Lecture 14 Handout: General Theory of n^{th} Order Linear Differential Equations

By Doron Zeilberger

Version of Dec. 11, 2016 (Thanks to Alex Eisenschmied)

The general form of a **linear** n^{th} **order linear diff.eq.** is (after you divide by the coefficient of $\frac{d^n y}{dt^n}$ (aka as $y^{(n)}(t)$)

$$\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y(t) = g(t) \quad ,$$

where, in general, the **coefficients**, $p_1(t), p_2(t), ..., p_n(t)$ and $g(t)$ are **functions** of t (not just *constants*).

In shorthand notation it is written

$$y^{(n)}(t) + p_1(t)y^{(n-1)}(t) + \ldots + p_{n-1}(t)y'(t) + p_n(t)y(t) = g(t) \quad .$$

Existence and Uniqueness Theorem for n^{th} Order Linear Diff.eqs.

If the coefficient-functions $p_1(t), p_2(t), \ldots p_n(t)$ and the right side, $g(t)$, are **continuous** (i.e. "well-behaved", they have no breaks and do not blow up) in an open interval I containing the number t_0, then we are **guaranteed** that there **exists** a **unique** solution $y(t)$ to the initial value diff.eq.

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t) \quad , \quad y(t_0) = z_0 \quad , \quad y'(t_0) = z_1 \quad , y^{(n-1)}(t_0) = z_{n-1} \quad ,$$

(for *any* numbers $z_0, z_1, \ldots, z_{n-1}$).

In other words, if the functions $p_1(t), p_2(t), \ldots, p_n(t)$,and $g(t)$ do not blow up, and have no surprises in I, and you want a function $y(t)$ that satisfies the diff.eq. satisfying the initial conditions then you are *promised* that there is such a function. Not only that, there is only **one** such function!

Problem 14.1: For each of the following diff.eq. initial value problems, and intervals, decide whether the theorem promises you that there is a unique solution.

a. $(t^2+1)\,y'''(t) + \sin t\, y''(t) + (t^3+1)\,y(t) = e^t \quad , \quad y(0) = 1 \quad , \quad y'(0) = 3 \qquad y''(0) = 5 \quad ; \quad -10 < t < 10$

b. $t^2\, y''''(t) + \sin t\, y''(t) + (t^3+1)\, y(t) = e^t \quad , \quad y(0) = 1 \quad , \quad y'(0) = 3 \quad , \quad y''(0) = 5 \quad , \quad y'''(0) = -3 \quad ; \quad -10 < t < 10$

Solution to 14.1a: We first **divide** by the coefficient of $y'''(t)$, t^2+1, getting the equivalent diff.eq. (initial value problem)

a.' $y'''(t) + \frac{\sin t}{t^2+1} y''(t) + \frac{t^3+1}{t^2+1} y(t) = \frac{e^t}{t^2+1}$

So $p_1(t) = \frac{\sin t}{t^2+1}$, $p_2(t) = 0$, $p_3(t) = \frac{t^3+1}{t^2+1}$, and $g(t) = \frac{e^t}{t^2+1}$. The numerators are all nice (continuous) functions, and the denominator in these, t^2+1 never vanishes in the interval $(-10, 10)$, so everything is nice, and the theorem promises a unique solution.

Solution to 14.1b: We first **divide** by the coefficient of $y'''(t)$, t^2, getting the equivalent diff.eq. (initial value problem)

b.' $y''''(t) + \frac{\sin t}{t^2} y''(t) + \frac{t^3+1}{t^2} y(t) = \frac{e^t}{t^2}$

So $p_1(t) = 0$ $p_2(t) = \frac{\sin t}{t^2}$, $p_3(t) = 0$ $p_4(t) = \frac{t^3+1}{t^2}$, and $g(t) = \frac{e^t}{t^2}$. The numerators are all nice (continuous) functions, **but** the denominator in these, t^2 vanishes at $t = 0$, and $t = 0$ happens to lie in the given interval, so (at least one of, but in this case all of) the coefficients blow up and hence are **not continuous**, and there is no guarantee.

Problem 14.2: Find the largest open interval for which the solution to

$(t-1)(t+3)\, y^{(4)}(t) + (\cos t)\, y''(t) + \sin t\, y'(t) + \cos t\, y(t) = t$,
$y(0) = 0, y'(0) = 1, y''(0) = -1, y'''(0) = 4,$

has a unique solution.

Solution to 14.2: After dividing by the coefficient of $y^{(4)}(t)$, $(t-1)(t+3)$, we get the initial value problem

$$y^{(4)}(t) + \frac{\cos t}{(t-1)(t+3)} y''(t) + \frac{\sin t}{(t-1)(t+3)}\, y'(t) \frac{\cos t}{(t-1)(t+3)} y(t) = \frac{t}{(t-1)(t+3)} .$$

The coefficients blow-up at $t = -3$ and $t = 1$. Our interval should contain $t_0 = 0$, so the largest interval is $-3 < t < 1$.

Ans. to 14.2: The largest interval is $-3 < t < 1$.

The amazing **Principle of Superposition** (only valid for **homogeneous** (linear) diff.eqs.!) that we already know is applicable for first and second-order linear homog. diff.eqs. is of course valid for any order!

If $y(x)$ is some solution, and c is a constant, then we know automatically that $cy(x)$ is also a solution.

If $y_1(x)$ and $y_2(x)$ are two solutions, then we know automatically that $y_1(x) + y_2(x)$ is also a solution.

Repeating this process, we know that if $y_1(x), \ldots, y_n(x)$ are *some* solutions of a homog. linear diff.eq. then any **linear combination**

$$y(x) \;=\; c_1 y_1(x) + \ldots + c_n y_n(x)$$

is also a solution. But, given a candidate set of n specific solutions $y_1(x), \ldots, y_n(x)$, can **any** solution $y(x)$ be expressed in terms of these proposed "atomic" functions?

There is beautiful criterion, called the Wronskian.

Important Theorem If the functions $p_1(t), \ldots, p_n(t)$ are continuous (i.e. don't blow up and don't have breaks) in an open interval I, and if the n functions $y_1(t), \ldots, y_n(t)$ are all solutions of the homog. linear diff.eq.

$$y^{(n)}(t) + p_1(t)y^{(n-1)}(t) + \ldots + p_{n-1}(t)y'(t) + p_n(t)y(t) = 0 \quad .$$

and the so-called Wronskian (see below), $W(y_1, y_2, \ldots, y_n)(t)$, is not always zero, then **Every** solution of of that diff.eq., $y(x)$, can be written as a *linear combination*

$$y(t) = c_1y_1(t) + \ldots + c_ny_n(t) \quad ,$$

for some **numbers** $c_1, \ldots, c_n$.

In that case we say that $y_1(t), y_2(t), \ldots, y_n(t)$ form a **fundamental set of solutions**.

Important Definition

The Wronskian determinant of a list of functions $y_1(t), \ldots, y_n(t)$ is:

$$\begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix}$$

Reminder about determinants

The determinant of a 2×2 matrix is defined by:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \quad .$$

The determinant of a 3×3 matrix is (if you expand with respect to the top row)

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} =$$

equals

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

Higher order determinants are computed analogously, but it is much more efficient to use *row operations* and *column operations* for larger determinants.

Problem 14.3: Decide whether the following functions are linearly independent or linearly dependent. In the latter case find a linear relation among them.

a. $y_1(t) = 1 + t$, $y_2(t) = 2 + t$, $y_3(t) = 3 + t$

b. $y_1(t) = 1 + t$, $y_2(t) = 1 + t^2$, $y_3(t) = 1 + t^3$

Solution to 14.3a: We are looking for numbers k_1, k_2, k_3 such that

$$k_1(1 + t) + k_2(2 + t) + k_3(3 + t) = 0 \quad ,$$

for all t. Simplifying, we get

$$(k_1 + k_2 + k_3)t + (k_1 + 2k_2 + 3k_3) = 0 \quad .$$

Setting the coefficients of t and 1 to 0 we get

$$k_1 + k_2 + k_3 = 0 \quad , \quad k_1 + 2k_2 + 3k_3 = 0 \quad .$$

From the first, we get $k_3 = -k_1 - k_2$. Plugging into the second, we get

$$k_1 + 2k_2 + 3(-k_1 - k_2) = 0 \quad ,$$

So

$$k_1 + 2k_2 - 3k_1 - 3k_2 = 0 \quad ,$$

$$-2k_1 - k_2 = 0 \quad ,$$

so $k_2 = -2k_1$ and going back to k_3: $k_3 = -k_1 - (-2k_1) = -k_1 + 2k_1 = k_1$. Taking, for example, $k_1 = 1$, we get

$$k_1 = 1 \quad , \quad k_2 = -2 \quad , \quad k_3 = 1 \quad .$$

So a linear relation is:

$$1 \cdot (1 + t) - 2 \cdot (2 + t) + 1 \cdot (3 + t) = 0 \quad .$$

Ans. to 14.3a: The three functions are **linearly dependent** and a linear relation is: $1 \cdot (1+t) - 2 \cdot (2 + t) + 1 \cdot (3 + t) = 0$.

Solution to 14.3b: We are looking for numbers k_1, k_2, k_3 such that

$$k_1(1 + t) + k_2(1 + t^2) + k_3(1 + t^3) = 0 \quad ,$$

for all t. Simplifying, we get

$$(k_1 + k_2 + k_3) + k_1 t + k_2 t^2 + k_3 t^3 = 0 \quad .$$

Setting the coefficients of t, t^2 and t^3 to zero, we get

$$k_1 = 0 \quad , \quad k_2 = 0 \quad , \quad k_3 = 0 \quad , \quad .$$

So the only solution is the **trivial** solution, and the three functions are **linearly independent**.

Ans. to 14.3b: The three functions are **linearly independent**.

Problem 4.4: Verify that the given functions are solutions of the diff.eq. and determine their Wronskian. **a.**

$$xy'''(x) - y''(x) = 0 \quad ; \quad 1 \quad , \quad x \quad , \quad x^3 .$$

b.

$$y'''(x) + y'(x) = 0 \quad ; \quad 1 \quad , \quad \sin x \quad , \quad \cos x$$

Solution to 4.4a: For $y(x) = 1$ we have

$$xy'''(x) - y''(x) = x1''' - 1'' = 0$$

For $y(x) = x$ we have

$$xy'''(x) - y''(x) = xx''' - x'' = 0$$

For $y(x) = x^3$ we have

$$xy'''(x) - y''(x) = x(x^3)''' - (x^3)'' = x(6) - 6x = 0 \quad ,$$

so there three functions are indeed solutions. Now the Wronskian is The determinant of a 3×3 matrix is (if you expand with respect to the top row)

$$\begin{vmatrix} 1 & x & x^3 \\ 0 & 1 & 3x^2 \\ 0 & 0 & 6x \end{vmatrix} = 6x \quad .$$

Answer to 4.4a: The three functions are indeed solutions and their Wronskian is $6x$.

Solution to 4.4b: For $y(x) = 1$ we have

$$y'''(x) + y'(x) = 1''' + 1' = 0$$

For $y(x) = \sin x$ we have $(y'(x) = \cos x, y''(x) = -\sin x, y'''(x) = -\cos x)$

$$y'''(x) + y'(x) = -\cos x + \cos x = 0$$

For $y(x) = \cos x$ we have $(y'(x) = -\sin x, y''(x) = -\cos x, y'''(x) = \sin x)$

$$y'''(x) + y'(x) = \sin x - \sin x = 0$$

so there three functions are indeed solutions. Now the Wronskian is The determinant of a 3×3 matrix is (if you expand with respect to the top row)

$$\begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix} = -\cos^2 x - \sin^2 x = -1 \quad .$$

Answer to 4.4b: The three functions are indeed solutions and their Wronskian is -1.