**Dr. Z.'s Calc4 Lecture 13 Handout: Variation of Parameters**

By Doron Zeilberger

**Important Theorem (Complicated Version)**

If the functions $p(t), q(t), g(t)$ are continuous on an open interval $I$, and if $y_1(t)$ and $y_2(t)$ are independent solutions of the **homogeneous** diff.eq.

$$y''(t) + p(t)\,y'(t) + q(t)\,y(t) = 0 \quad ,$$

then a particular solution of the **inhomogeneous** diff.eq.

$$y''(t) + p(t)\,y'(t) + q(t)\,y(t) = g(t) \quad ,$$

is given by

$$-y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1,y_2)(s)}\,ds \;+\; y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1,y_2)(s)}\,ds$$

where $W(y_1,y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$.

**Important Theorem (Simple Version)**

If the functions $p(t), q(t), g(t)$ are continuous on an open interval $I$, and if $y_1(t)$ and $y_2(t)$ are independent solutions of the **homogeneous** diff.eq.

$$y''(t) + p(t)\,y'(t) + q(t)\,y(t) = 0 \quad ,$$

then a particular solution of the **inhomogeneous** diff.eq.

$$y''(t) + p(t)\,y'(t) + q(t)\,y(t) = g(t) \quad ,$$

is given by

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where $u_1(t), u_2(t)$ are two functions whose derivatives satisfy the system of two equations

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0 \quad ,$$

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t) \quad ,$$

**Problem 13.1**: **Using Variation of Parameters**, find a particular solution of

$$y''(t) - y'(t) - 2y(t) = 2e^{-t} \quad .$$

**Solution of 13.1**: The characteristic equation of the homog. version is $r^2 - r - 2 = 0$. Factoring $(r-2)(r+1) = 0$, whose roots are $r = 2, r = -1$, so

$$y_1(t) = e^{-t} \quad , \quad y_2(t) = e^{2t} \quad .$$

We also need the derivatives

$$y_1'(t) = -e^{-t} \quad , \quad y_2'(t) = 2e^{2t} \quad .$$

The function $g(t)$ is the **right hand side** (after we have divided by the coefficient of $y''(t)$, in this case it is 1), so $g(t) = 2e^{-t}$.

We are looking for two functions $u_1'(t)$ and $u_2'(t)$ such that

$$u_1'(t)e^{-t} + u_2'(t)e^{2t} = 0 \quad ,$$

$$u_1'(t)(-e^{-t}) + u_2'(t)(2e^{2t}) = 2e^{-t} \quad ,$$

Cleaning up (multiplying by $e^t$)

$$u_1'(t) + u_2'(t)e^{3t} = 0 \quad ,$$

$$-u_1'(t) + 2u_2'(t)e^{3t} = 2 \quad ,$$

From the first equation, we get

$$u_1'(t) = -e^{3t}u_2'(t) \quad .$$

Pluging into the second

$$e^{3t}u_2'(t) + 2e^{3t}u_2'(t) = 2 \quad .$$

Collecting terms

$$3e^{3t}u_2'(t) = 2 \quad .$$

Dividing by $3e^{3t}$:

$$u_2'(t) = \frac{2}{3}e^{-3t} \quad .$$

Going back to $u_1'(t)$:

$$u_1'(t) = -e^{3t}\frac{2}{3}e^{-3t} = -\frac{2}{3} \quad .$$

So we have

$$u_1'(t) = -\frac{2}{3} \quad , \quad u_2'(t) = \frac{2}{3}e^{-3t} \quad .$$

Integrating (we don't have to worry about the $+C$)

$$u_1(t) = -\frac{2}{3}t \quad , \quad u_2(t) = -\frac{2}{9}e^{-3t} \quad .$$

**Finally**, we plug these into

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

So

$$Y(t) = (-\frac{2}{3}t)e^{-t} - \frac{2}{9}e^{-3t}e^{2t} = -\frac{2}{3}te^{-t} - \frac{2}{9}e^{-t} \quad .$$

**First Answer to 13.1**: A particular solution is $Y(t) = -\frac{2}{3}te^{-t} - \frac{2}{9}e^{-t}$.

But since the second term is a multiple of $y_1(t)$ and adding or subtracting any constant multipe of $y_1(t)$ and/or $y_2(t)$ from a particular solution is still (another, possibly simpler) particular solution, we can forget about the second term and get

**Second Answer to 13.1**: An even better particular solution is $Y(t) = -\frac{2}{3}te^{-t}$.

**Problem 13.2**: **Using Variation of Parameters**, find a particular solution of

$$y''(t) - 2y'(t) + y(t) = \frac{e^t}{1+t^2} \quad .$$

**Solution of 13.2**: The characteristic equation of the homog. version is $r^2 - 2r + 1 = 0$. Factoring $(r-1)^2 = 0$, and there is a **double root**, $r = 1$. So

$$y_1(t) = e^t \quad , \quad y_2(t) = te^t \quad .$$

We also need the derivatives

$$y_1'(t) = e^t \quad , \quad y_2'(t) = (t+1)e^t \quad .$$

The function $g(t)$ is the **right hand side** (after we have divided by the coefficient of $y''(t)$, in this case it is 1), so $g(t) = \frac{e^t}{t^2+1}$.

We are looking for two functions $u_1'(t)$ and $u_2'(t)$ such that

$$u_1'(t)e^t + u_2'(t)te^t = 0 \quad ,$$

$$u_1'(t)e^t + u_2'(t)(t+1)e^t = \frac{e^t}{1+t^2} \quad ,$$

Cleaning up (dividing by $e^t$)

$$u_1'(t) + u_2'(t)t = 0 \quad ,$$

$$u_1'(t) + (t+1)u_2'(t) = \frac{1}{1+t^2} \quad .$$

From the first equation, we get

$$u_1'(t) = -tu_2'(t) \quad .$$

Pluging into the second

$$-tu_2'(t) + (t+1)u_2'(t) = \frac{1}{1+t^2} \quad ,$$

Simplifying:

$$u_2'(t) = \frac{1}{1+t^2} \quad .$$

Going back to $u_1'(t)$:

$$u_1'(t) = -tu_2'(t) = -\frac{t}{1+t^2}$$

So we have

$$u_1'(t) = -\frac{t}{1+t^2} \quad , \quad u_2'(t) = \frac{1}{1+t^2} \quad .$$

Integrating (we don't have to worry about the $+C$)

$$u_1(t) = -\frac{1}{2}\ln(1+t^2) \quad , \quad u_2(t) = \arctan t \quad .$$

**Finally**, we plug these into

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

So

$$Y(t) = -\frac{1}{2}e^t \ln(1+t^2) + te^t \arctan t$$

**Answer to 13.2**: A particular solution is $Y(t) = -\frac{1}{2}e^t \ln(1+t^2) + te^t \arctan t$.

**Problem 13.3**: Verify that the given functions $y_1(x)$, $y_2(x)$ are solutions of the corresponding homogeneous linear diff.eq., and find the general solution of the diff.eq.

$$x^2y''(x) - 3xy'(x) + 4y(x) = x^2 \ln x \quad , \quad x > 0 \quad ; \quad y_1(x) = x^2 \quad , \quad y_2(x) = x^2 \ln x \quad .$$

**Solution of 13.3**: $y_1(x) = x^2 \quad , \quad y_1'(x) = 2x \quad , \quad y_1''(x) = 2$, so

$$x^2y_1''(x) - 3xy_1'(x) + 4y_1(x) = x^2(2) - 3x(2x) + 4x^2 = 2x^2 - 6x^2 + 4x^2 = 0 \quad .$$

Also

$y_2(x) = x^2 \ln x \quad , \quad y_2'(x) = 2x \ln x + x \quad , \quad y_2''(x) = 2\ln x + 2 + 1 = 2\ln x + 3$, so

$$x^2y_2''(x) - 3xy_2'(x) + 4y_2(x) = x^2(2\ln x + 3) - 3x(2x\ln x + x) + 4x^2 \ln x = 0 \quad .$$

So both $y_1(x) = x^2$ and $y_2(x) = x^2 \ln x$ are indeed solutions of the homogeneous version.

The function $g(x)$ is the **right hand side** after we have divided by the coefficient of $y''(t)$, so $g(x) = \ln x$ , We are looking for two functions $u_1'(x)$ and $u_2'(x)$ such that

$$u_1'(x)\,x^2 + u_2'(x)\,x^2\ln x = 0 \quad ,$$

$$u_1'(x)\,(2x) + u_2'(x)\,(2x\ln x + x) = \;lnx \quad ,$$

From the first equation

$$u_1'(x) = -(\ln x)\,u_2'(x) \quad .$$

Pluging into the second

$$-(\ln x\,u_2'(x))\,(2x) + u_2'(x)\,(2x\ln x + x) = \ln x$$

Simplifying:

$$u_2'(x) = x^{-1}\ln x \quad .$$

Going back to $u_1'(x)$:

$$u_1'(x) = -(\ln x)^2 x^{-1}$$

So we have

$$u_1'(x) = -(\ln x)^2 x^{-1} \quad , \quad u_2'(x) = (\ln x)x^{-1} \quad .$$

Integrating (we don't have to worry about the $+C$)

$$u_1(x) = -\frac{1}{3}(\ln x)^3 \quad , \quad u_2(x) = \frac{1}{2}(\ln x)^2 \quad .$$

**Finally**, we plug these into

$$Y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

So a particular solution is

$$Y(x) = -\frac{1}{3}(\ln x)^3(x^2) + \frac{1}{2}(\ln x)^2(x^2\ln x) = (\frac{1}{2} - \frac{1}{3})x^2(\ln x)^3 = \frac{1}{6}x^2(\ln x)^3 \quad .$$

So, a particular solution is $Y(x) = \frac{1}{6}x^2(\ln x)^3$.

**Finally Finally**, to get the **general solution** of the diff.eq. we add the general solution of the homogeneous version $c_1y_1(x) + c_2y_2(x)$, which in this problem is $c_1x^2 + c_2x^2\ln x$.

**Answer to 13.3**: The general solution of the diff.eq. is $y(x) = c_1x^2 + c_2x^2\ln x + \frac{1}{6}x^2(\ln x)^3$.