Dr. Z.'s Calc4 Lecture 12 Handout:
NonHomogeneous (Linear) Second-Order Diff. Eqs.; Undetermined Coefficients

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In order to find the **general solution** of a (general) linear second-order diff.eq.

$$y''(t) + p(t)\, y'(t) + q(t)\, y(t) = g(t) \quad ,$$

you find the general solution of the **corresponding** *homogeneous* version (obtained by replacing $g(t)$ by 0)

$$y''(t) + p(t)\, y'(t) + q(t)\, y(t) = 0 \quad ,$$

that has the format $c_1 y_1(t) + c_2 y_2(t)$ for some specific solutions $y_1(t), y_2(t)$ (whose Wronskian is not identically zero).

Then by **hook or crook** you find **ONE** specific solution P(t) (called the **particular solution**) of the original (inhomogeneous diff.eq.). Then

$$General \quad Solution = c_1 y_1(t) + c_2 y_2(t) + P(t) \quad .$$

In other words

General Solution of InHomog. Diff.Eq.= General Solution of Homog. Diff.Eq. + Particular Solution .

How to find Particular Solutions of Constant Coefficient Inhomog. linear diff.eqs.

If the right side is a (specific) polynomial of degree k, your **first try** is a generic polynomial (with undetermined coefficients) of degree k. In particular if it is a constant, you try out $y(t) = A_0$.

If the right side is a (specific) exponential, $e^{\alpha t}$, your **first try** is a $A_0 e^{\alpha t}$.

If the right side is a (specific) exponential times trig, $e^{\alpha t} \sin \beta t$, or $e^{\alpha t} \cos \beta t$, or, more generally, $e^{\alpha t}(A \sin \beta t + B \cos \beta t)$ your **first try** is a $e^{\alpha t}(A_0 \sin \beta t + B_0 \cos \beta t)$.

If the right side is a (specific) exponential, times a specific polynomial of degree k, $e^{\alpha t} p(t)$, your **first try** is a $e^{\alpha t}(A_0 t^k + A_1 t^{k-1} + \ldots + A_k t^0)$.

If the **first try** does not work, multiply the proposed Particular Solution by t.

Problem 12.1: Find a particular solution of the diff.eq.

$$y''(t) + 3\, y'(t) + 2\, y(t) = 6$$

Solution of 12.1: Since the right side is a **constant**, you try out

$$y(t) = A_0 \quad ,$$

where A_0 is a constant yet **to be determined**.

Now $y'(t) = 0, y''(t) = 0$ so, plugging into the diff.eq.

$$0 + 3 \cdot 0 + 2 \cdot A_0 = 6$$

Solving the equation $2A_0 = 6$ we get $A_0 = 3$. So a particular solution is $y(t) = 3$.

Ans. to 12.1: a particular solution is $y(t) = 3$.

Problem 12.2: Find a particular solution of the diff.eq.

$$y''(t) + 3\,y'(t) + 2\,y(t) = 2t^2 + 8t + 7$$

Solution of 12.2: Since the right side is a polynomial in t of **degree 2**, you try out

$$y(t) = A_0 t^2 + A_1 t + A_2$$

where A_0, A_1, A_2 are constants yet **to be determined**.

Now $y'(t) = 2A_0 t + A_1, y''(t) = 2A_0$ so, plugging into the diff.eq.

$$2A_0 + 3(2A_0 t + A_1) + 2(A_0 t^2 + A_1 t + A_2) = 2t^2 + 8t + 7 \quad .$$

Simplifying:

$$(2A_0)t^2 + (2A_1 + 6A_0)t + (2A_0 + 3A_1 + 2A_2) = 2t^2 + 8t + 7 \quad .$$

We now *compare coefficients* of t^2, t^1, t^0 on both sides getting *three* linear equations for the *three* unknowns A_0, A_1, A_2.

$$2A_0 = 2 \quad , \quad 2A_1 + 6A_0 = 8 \quad , \quad 2A_0 + 3A_1 + 2A_2 = 7 \quad .$$

Solving these equation, we get from the first, $A_0 = 1$, then from the second $2A_1 + 6 \cdot 1 = 8$, so $A_1 = 1$, and from the from the third $2(1) + 3(1) + 2A_2 = 7$, so $2A_2 = 2$ and $A_2 = 1$.

Going back to the general template $A_0 t^2 + A_1 t + A_2$ we get that a particular solution is $y(t) = t^2 + t + 1$.

Ans. to 12.2: a particular solution is $y(t) = t^2 + t + 1$.

Problem 12.3: Find a particular solution of the diff.eq.

$$y''(t) + 3\,y'(t) + 2\,y(t) = e^{2t}$$

Solution of 12.3: Since the right side is a pure **exponential function** e^{2t}, we simply try

$$y(t) = A_0 e^{2t}$$

where A_0 is a constant yet **to be determined**.

Now $y'(t) = 2A_0e^{2t}, y''(t) = 4A_0e^{2t}$ so, plugging into the diff.eq.

$$4A_0e^{2t} + 3(2A_0e^{2t}) + 2(A_0e^{2t}) = e^{2t} \quad .$$

Simplifying:

$$(4A_0 + 6A_0 + 2A_0)e^{2t} = e^{2t}$$

$$12A_0e^{2t} = e^{2t} \quad .$$

Dividing by e^{2t}, $12A_0 = 1$, and solving for A_0, we get $A_0 = \frac{1}{12}$.

Going back to the general template A_0e^{2t} we get that a particular solution is $y(t) = \frac{1}{12}e^{2t}$.

Ans. to 12.3: a particular solution is $y(t) = \frac{1}{12}e^{2t}$.

Problem 12.4: Find a particular solution of the diff.eq.

$$y''(t) + 3\,y'(t) + 2\,y(t) = (12t + 7)e^{2t}$$

Solution of 12.4: Since the right side is a pure **exponential function** e^{2t}, **times** a certain polynomial of degree 1, we try

$$y(t) = (A_0t + A_1)e^{2t}$$

where A_0, A_1 are constants yet **to be determined**.

Now $y'(t) = (2A_0t + (2A_1 + A_0))e^{2t}$, $y''(t) = ((4A_0t + (4A_1 + 4A_0))e^{2t}$, so, plugging into the diff.eq.

$$((4A_0t + (4A_1 + 4A_0))e^{2t} + 3((2A_0t + (2A_1 + A_0)) + 2(A_0t + A_1)e^{2t} = (6t + 11)e^{2t}$$

Simplifying:

$$(12A_0t + (12A_1 + 7A_0))e^{2t} = (12t + 7)e^{2t}$$

Dividing by e^{2t}:

$$12A_0t + (12A_1 + 7A_0) = 12t + 7$$

Comparing coefficients:

$$12A_0 = 12 \quad , \quad 12A_1 + 7A_0 = 7 \quad .$$

From the first equation: $A_0 = 1$ from the second $12A_1 + 7 = 7$, so $A_1 = 0$.

Going back to the general template $(A_0t + A_1)e^{2t}$ we get that a particular solution is $y(t) = te^{2t}$.

Ans. to 12.4: a particular solution is $y(t) = te^{2t}$.

Problem 12.5: Find a particular solution of the diff.eq.

$$y''(t) + 3\,y'(t) + 2\,y(t) = e^{-t}$$

Solution of 12.5: Since the right side is a pure **exponential function** e^{-t}, the **first try** is

$$y(t) = A_0 e^{-t} \quad .$$

Now

$$y'(t) = -A_0 e^{-t} \quad , \quad y''(t) = A_0 e^{-t} \quad .$$

Plugging into the diff.eq.

$$A_0 e^{-t} + 3(-A_0 e^{-t}) + 2A_0 e^{-2t} = e^{-t}$$

Simplifying, we get:

$$0 = e^{-t} \quad .$$

NONSENSE!. So the first try failed. We now have a **second try**, by multiplying the first try by t.

$$y(t) = A_0 t e^{-t} \quad .$$

Now

$$y'(t) = A_0(1-t)e^{-t} \quad , \quad y''(t) = A_0(t-2)e^{-t} \quad .$$

Plugging into the diff.eq.

$$A_0(t-2)e^{-t} + 3(1-t)A_0 e^{-t} + 2A_0 e^{-t} = e^{-t}$$

Simplifying

$$A_0((t-2) + 3(1-t) + 2t)e^{-t} = e^{-t}$$

$$A_0(t - 2 + 3 - 3t + 2t)e^{-t} = e^{-t}$$

$$A_0 e^{-t} = e^{-t}$$

Dividing by e^{-2t}, we get

$$A_0 = 1 \quad .$$

Going back to the template, we get that a particular solution is $y(t) = te^{-t}$.

Problem 12.1': Find the general solution of the diff.eq.

$$y''(t) + 3\,y'(t) + 2\,y(t) = 6$$

Solution of 12.1: We first find the general solution of the **homogeneous version**

$$y''(t) + 3\,y'(t) + 2\,y(t) = 0$$

The **characteristic equation** is $r^2 + 3r + 2 = 0$, factorizing $(r+1)(r+2) = 0$, so the roots are $r_1 = -1, r_2 = -2$, and the general solution of the homog. version is

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} \quad .$$

We now do the Particular Solution (see Problem 12.1), and get P.S. $y(t) = 3$. We now **Add them up** and get that the **general solution** of our diff.eq. is:

$$y(t) = 3 + c_1 e^{-t} + c_2 e^{-2t} \quad .$$

Ans. to 12.1': $y(t) = 3 + c_1 e^{-t} + c_2 e^{-2t}$

Problem 12.5': Find the general solution of the diff.eq.

$$y''(t) + 3\,y'(t) + 2\,y(t) = e^{-t}$$

Solution of 12.5': We first find the general solution of the **homogeneous version**

$$y''(t) + 3\,y'(t) + 2\,y(t) = 0$$

The **characteristic equation** is $r^2 + 3r + 2 = 0$, factorizing $(r+1)(r+2) = 0$, so the roots are $r_1 = -1, r_2 = -2$, and the general solution of the homog. version is

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} \quad .$$

We now do the Particular Solution (see Problem 12.5), and get P.S. $y(t) = te^{-t}$. We now **Add them up** and get that the **general solution** of our diff.eq. is:

$$y(t) = te^{-t} + c_1 e^{-t} + c_2 e^{-2t} \quad .$$

Ans. to 12.5': $y(t) = te^{-t} + c_1 e^{-t} + c_2 e^{-2t}$

Problem 12.1": Solve the initial value problem

$$y''(t) + 3\,y'(t) + 2\,y(t) = 6 \quad , y(0) = 6 \quad , \quad y'(0) = -4$$

Solution of 12.1": We first do 12.1', getting the general solution

$$y(t) = 3 + c_1 e^{-t} + c_2 e^{-2t} \quad .$$

Now

$$y'(t) = -c_1 e^{-t} - 2c_2 e^{-2t} \quad .$$

So

$$y(0) = 3 + c_1 e^{-0} + c_2 e^{-2\cdot 0} = 3 + c_1 + c + 2 \quad .$$

$$y'(0) = -c_1 e^{-0} - 2c_2 e^{-2\cdot 0} = -c_1 - 2c_2$$

Using our initial condition, we have to solve

$$3 + c_1 + c_2 = 6 \quad , \quad -c_1 - 2c_2 = -4$$

i.e.

$$c_1 + c_2 = 3 \quad , \quad c_1 + 2c_2 = 4$$

Subtracting the second equation from the first, we get $c_2 = 1$ and plugging this into the first (or second) equation we get $c_1 = 2$. Going back to the general solution, we get the solution is

$$y(t) = 3 + 2e^{-t} + e^{-2t} \quad .$$

Ans. to 12.1": $y(t) = 3 + 2e^{-t} + e^{-2t}$.

Problem 12.5": Solve the initial value problem

$$y''(t) + 3\,y'(t) + 2\,y(t) = e^{-t} \quad , y(0) = 1 \quad , \quad y'(0) = -1$$

Solution of 12.5": We first do 12.5', getting the general solution

$$y(t) = te^{-t} + c_1 e^{-t} + c_2 e^{-2t} \quad .$$

Now

$$y'(t) = (1-t)e^{-t} - c_1 e^{-t} - 2c_2 e^{-2t} \quad .$$

Plugging-in $t = 0$:

$$y(0) = 0 \cdot e^{-0} + c_1 e^{-0} + c_2 e^{-2\cdot 0} = c_1 + c_2 \quad .$$

$$y'(0) = (1-0)e^{-0} - c_1 e^{-0} - 2c_2 e^{-2\cdot 0} = 1 - c_1 - 2c_2 \quad .$$

So

$$c_1 + c_2 = 1 \quad , \quad 1 - c_1 - 2c_2 = -1$$

Simplifying

$$c_1 + c_2 = 1 \quad , \quad c_1 + 2c_2 = 2$$

Subracting the second equation from the first, we get $c_2 = 1$, $c_1 = 0$. Going back to the general solution we have

$$y(t) = te^{-t} + 0 \cdot e^{-t} + 1 \cdot e^{-2t} = te^{-t} + e^{-2t} \quad .$$

Ans. to 12.5" $y(t) = te^{-t} + e^{-2t}$.