Dr. Z.'s Calc4 Lecture 11 Handout: Solving Second-Order Homogeneous Linear Diff.Eqs. With Constant Coefficients When the Characteristic Equation Has Repeated Roots and Reduction of Order

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Recall that if we have a **constant coefficient** second-order linear homog. diff.eq. of the form

$$a\,y''(t) + b\,y'(t) + c\,y(t) = 0$$

we form the **characteristic equation**, that happens to be a (simple algebraic, quadratic) equation

$$ar^2 + br + c = 0 \quad ,$$

obtained by replacing $y''(t)$ by r^2, $y'(t)$ by r and $y(t)$ by 1.

We then solve it either by factorization, or via the famous formula

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad .$$

And the general solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \quad .$$

But what if we have **repeated roots**?, i.e. what if the **discriminant**, $b^2 - 4ac$ is **zero**?

Then we only have **one** root, that is a **double root**

$$r_1 = -\frac{b}{2a} \quad .$$

Important Theorem: If $b^2 - 4ac = 0$, then the general solution of the diff.eq.

$$a\,y''(t) + b\,y'(t) + c\,y(t) = 0$$

whose double root is $r_1 = -\frac{b}{2a}$ is given by

$$y(t) = e^{r_1 t}(c_1 + c_2\,t) \quad .$$

Problem 11.1: Find the general solution to the diff.eq.

$$y''(t) - 6\,y'(t) + 9\,y(t) = 0 \quad .$$

Solution of 11.1: The **characteristic equation** is

$$r^2 - 6r + 9 = 0 \quad .$$

Factorizing, we get

$$(r-3)^2 = 0 \quad ,$$

so we have a **repeated root** $r_1 = 3$. So the general solution is:

$$y(t) = e^{3t}(c_1 + c_2 t) \quad .$$

Ans. to 11.1: $y(t) = e^{3t}(c_1 + c_2\, t)$ (or $y(t) = c_1 e^{3t} + c_2 t e^{3t}$).

Problem 11.2: Find the solution of the initial value problem

$$y''(t) + 4y'(t) + 4y(t) = 0 \quad , \quad y(0) = 1 \quad , \quad y'(0) = 0 \quad .$$

Solution to 11.2: We first find the **general solution**. The characteristic equation is:

$$r^2 + 4r + 4 = 0 \quad .$$

Factorizing, we get $(r+2)^2 = 0$, so so we have a **repeated root** $r_1 = -2$. The general solution is

$$y(t) = e^{-2t}(c_1 + c_2\, t) \quad .$$

Next we find an expression for $y'(t)$:

$$y'(t) = e^{-2t}(c_1 + c_2\, t)' + (e^{-2t})'(c_1 + c_2\, t)$$

$$= e^{-2t}(c_2) - 2e^{-2t}(c_1 + c_2\, t)$$

$$= e^{-2t}(c_2 - 2c_1) - 2c_2 t\, e^{-2t} \quad .$$

Now we plug-in $t = 0$.

$$y(0) = e^{-2\cdot 0}(c_1 + c_2 \cdot 0) = c_1 \quad .$$

$$y'(0) = e^{-2\cdot 0}(c_2 - 2c_1) - 2c_2 \cdot 0 \cdot\, e^{-2\cdot 0} = c_2 - 2c_1$$

Using the initial conditions $y(0) = 1 \quad , \quad y'(0) = 0$ we have to solve the set of two equations

$$c_1 = 1 \quad , \quad c_2 - 2c_1 = 0 \quad .$$

From the first we get $c_1 = 1$. Plugging into the second: $c_2 - 2 \cdot 1 = 0$ so $c_2 = 2$.

Finally we go back to the above general solution and replace c_1 by 1 and c_2 by 2 getting that

$$y(t) = e^{-2t}(1 + 2\, t) \quad .$$

Ans. to 10.2: $y(t) = e^{-2t}(1 + 2\, t)$ (or $y(t) = e^{-2t} + 2te^{-2t}$).

Reduction of Order

Suppose that you have a *general* second-order homog. linear diff.eq. (not (necessarily) with constant coefficients), that can always be written (by dividing by the coefficient of $y''(t)$, if it is not already 1):

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0 \quad .$$

Suppose that someone gave you **one** solution, $y_1(t)$, then you can find another (independent!) solution, $y_2(t)$ (and hence **all** solutions, since the general solution would be $c_1y_1(t) + c_2y_2(t)$), by writing

$$y(t) = v(t)y_1(t) \quad ,$$

where the function $v(t)$ is **to be determined**.

Of course, once you have found $v(t)$, you would have $y(t)$, that you would call $y_2(t)$.

After some manipulations, it can be seen that v satisfies the diff.eq.

$$y_1(t)v''(t) + (2y_1'(t) + p(t)y_1(t))v'(t) = 0 \quad .$$

This is a **first** order diff.eq. for $v'(t)$. Having found $v'(t)$, you get $v(t)$ by integrating.

Warning: Do not memorize this. It would be part of the formula sheet on the exam.

Note: If you know how to integrate, then you know how to solve first-order linear diff.eq. Unfortunately, most integrals not even Maple (or God) can do in closed-form. In that case you should leave your answer in terms of integral signs.

Problem 11.3: Verify that the given solution $y_1(t)$ is indeed a solution of the given diff.eq. . Then Find a second solution of the given differential equation. Then write down the general solution.

$$t^2y''(t) + 3ty'(t) + y(t) = 0 \quad , \quad t > 0 \quad ; \quad y_1(t) = \frac{1}{t} \quad .$$

Solution of 11.3:

Step 1. For $y(t) = 1/t$, $y(t) = t^{-1}, y'(t) = (-1)t^{-2} = -t^{-2}, y''(t) = 2t^{-3}$, so

$$t^2y''(t) + 3ty'(t) + y(t) = t^2(2t^{-3}) + 3t(-t^{-2}) + t^{-1} = 2t^{-1} - 3t^{-1} + t^{-1} = 0 \quad ,$$

so $y_1(t) = t^{-1}$ is indeed **a** solution.

Step 2: Divide by the coefficient of $y''(t)$ (if it is not already 1), to get it in standard form.

$$y''(t) + \frac{3t}{t^2}y'(t) + \frac{1}{t^2}y(t) = 0$$

$$y''(t) + \frac{3}{t}y'(t) + \frac{1}{t^2}y(t) = 0 \quad .$$

Look at the coefficient of $y'(t)$, to find out $p(t)$.

$$p(t) = \frac{3}{t} \quad .$$

Step 3: Incorpoate the above $p(t)$ and $y_1(t)$ into the general template (that you would not need to memorize, it would be given to you).

$$y_1(t)v''(t) + (2y_1'(t) + p(t)y_1(t))v'(t) = 0$$

Set up the diff.eq. for $v(t)$.

Here $p(t) = \frac{3}{t}$, $y_1(t) = t^{-1}$, $y_1'(t) = -t^{-2}$. So in this problem:

$$t^{-1}v''(t) + (2(-t^{-2}) + 3t^{-1}t^{-1})v'(t) = 0$$

Simplifying:

$$t^{-1}v''(t) + (-2t^{-2} + 3t^{-2})v'(t) = 0$$

$$t^{-1}v''(t) + t^{-2}v'(t) = 0$$

$$v''(t) + \frac{1}{t}v'(t) = 0 \quad .$$

Step 4 Rename $v'(t)$, $u(t)$, write down the first-order diff.eq. for $u(t)$, and use the method of **Integrating factor** (or any other method) to solve it.

$$u'(t) + \frac{1}{t}u(t) = 0 \quad .$$

Here $p(t) = t^{-1}$ (note: here $p(t)$ means something else). Integrating factor is

$$I(t) = exp(\int p(t)\,dt) = exp(\int \frac{1}{t}\,dt) = exp(\ln t) = t \quad .$$

Multiplying by $I(t) = t$

$$tu'(t) + u(t) = 0 \quad .$$

Product rule:

$$(tu(t))' = 0$$

Integrating

$$tu(t) = C \quad ,$$

so $u(t) = \frac{C}{t}$. Since we are only interested in **one** solution (for now), we take $C = 1$ and get

$$u(t) = \frac{1}{t}$$

Step 5. To get $v(t)$, integrate $u(t)$

$$v(t) = \int \frac{1}{t}\,dt = \ln t \quad .$$

Step 6. Go back to $y_2(t) = v(t)y_1(t)$, and write down the **new** solution to the diff.eq.

$$y_2(t) = t^{-1} \ln t \quad .$$

Second Part of Answer of 11.3: $y_2(t) = t^{-1} \ln t$.

Step 7: Write down the general solution $c_1 y_1(t) + c_2 y_2(t)$

$$y(t) = c_1 t^{-1} + c_2 t^{-1} \ln t \quad .$$

Third Part of Answer of 11.3: $y(t) = c_1 t^{-1} + c_2 t^{-1} \ln t$ or $y(t) = \frac{c_1 + c_2 \ln t}{t}$.