Dr. Z.'s Calc4 Lecture 10 Handout: Solving Second-Order Homogeneous Linear Diff.Eqs. With Constant Coefficients When the Characteristic Equation Has Complex Roots

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Recall that if we have a **constant coefficient** second-order linear homog. diff.eq. of the form

$$a\,y''(t) + b\,y'(t) + c\,y(t) = 0$$

we form the **characteristic equation**, that happens to be a (simple algebraic, quadratic) equation

$$ar^2 + br + c = 0 \quad ,$$

obtained by replacing $y''(t)$ by r^2, $y'(t)$ by r and $y(t)$ by 1.

We then solve it either by factorization, or via the famous formula

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad .$$

And the general solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \quad .$$

But what if we have **complex roots**?, i.e. what if the **discriminant**, $b^2 - 4ac$ is **negative**?

Then we write

$$r_1, r_2 = \lambda \pm i\mu \quad ,$$

and the above formula, strictly speaking, is still applicable. But since we live in the *real world*, we use Euler's famous

$$e^{iz} = \cos z + i \sin z \quad .$$

After some rearranging

Important Theorem: If $b^2 - 4ac < 0$, then the general solution of the diff.eq.

$$a\,y''(t) + b\,y'(t) + c\,y(t) = 0$$

whose roots are $\lambda \pm i\mu$ is given by

$$y(t) = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t) \quad .$$

Problem 10.1: Find the general solution to the diff.eq.

$$y''(t) - 2y'(t) + 6y(t) = 0 \quad .$$

Solution of 10.1: The **characteristic equation** is

$$r^2 - 2r + 6 = 0 \quad .$$

The roots are

$$\frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(6)}}{2} = \frac{2 \pm \sqrt{-20}}{2} = \frac{2 \pm i\sqrt{4 \cdot 5}}{2} = \frac{2 \pm i2\sqrt{5}}{2} = 1 \pm i\sqrt{5} \quad .$$

Here $\lambda = 1$, $\mu = \sqrt{5}$. So the general solution is

$$y(t) = e^t(c_1 \cos \sqrt{5}t + c_2 \sin \sqrt{5}t) \quad .$$

Ans. to 10.1: $y(t) = e^t(c_1 \cos \sqrt{5}t + c_2 \sin \sqrt{5}t)$ (or $y(t) = c_1 e^t \cos \sqrt{5}t + c_2 e^t \sin \sqrt{5}t$).

Note:

If $\lambda = 0$ then we have *steady oscillation*

If $\lambda > 0$ then we have *growing oscillation*

If $\lambda < 0$ then we have *decaying oscillation*

Problem 10.2: Find the solution of the initial value problem

$$y''(t) + 4y'(t) + 5y(t) = 0 \quad , \quad y(0) = 1 \quad , \quad y'(0) = 0 \quad .$$

What kind of oscillation is it?

Solution to 10.2: We first find the **general solution**. The characteristic equation is:

$$r^2 + 4r + 5 = 0 \quad .$$

Solving it:

$$r_1, r_2 = \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i \quad .$$

So $\lambda = -2$, $\mu = 1$.

The general solution is

$$y(t) = e^{-2t}(c_1 \cos t + c_2 \sin t) \quad .$$

Next we find an expression for $y'(t)$:

$$y'(t) = e^{-2t}(c_1 \cos t + c_2 \sin t)' + (e^{-2t})'(c_1 \cos t + c_2 \sin t)$$

$$= e^{-2t}(-c_1 \sin t + c_2 \cos t) - 2e^{-2t}(c_1 \cos t + c_2 \sin t)$$

$$= e^{-2t}(-c_1 \sin t + c_2 \cos t - 2c_1 \cos t - 2c_2 \sin t)$$

$$= e^{-2t}(-(c_1 + 2c_2) \sin t + (c_2 - 2c_1) \cos t) \quad .$$

Now we plug-in $t = 0$.

$$y(0) = e^{-2\cdot 0}(c_1 \cos 0 + c_2 \sin 0) = c_1$$

$$y'(0) = e^{-2\cdot 0}(-(c_1 + 2c_2) \sin 0 + (c_2 - 2c_1) \cos 0) = c_2 - 2c_1 \quad .$$

Using the initial conditions $y(0) = 1 \quad , \quad y'(0) = 0$ we have to solve the set of two equations

$$c_1 = 1 \quad , \quad c_2 - 2c_1 = 0 \quad .$$

From the first we get $c_1 = 1$. Plugging into the second: $c_2 - 2 \cdot 1 = 0$ so $c_2 = 2$.

Finally we go back to the above general solution and replace c_1 by 1 and c_2 by 2 getting that

$$y(t) = e^{-2t}(\cos t + 2 \sin t)$$

Since -2 is **negative** it is **decaying oscillation**.

Ans. to 10.2: $y(t) = e^{-2t}(\cos t + 2 \sin t)$ (or $y(t) = e^{-2t} \cos t + 2e^{-2t} \sin t$). Decaying oscillation.