CORRECTING TYPOS and ERRORS POINTED OUT BY Aastha and AAsyushi Kasera

NAME: (print!) Dr. Z.

Section: ALL **E-Mail address:** DrZcalc3@gmail.com

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Makeup for Exam 1, Sunday, Oct. 25, 2020, 9:00-11:00am

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WRITE YOUR FINAL ANSWERS BELOW

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1. $(t, 0, 0)$, $-\infty < t < \infty$; x-axis .

2. 272

3. $(-3, 3)$.

4. $z = 2x + 12y - 17$

5. $\frac{1}{2}$

6. $\frac{24999}{5000}$ or 4.9998

7. DNE

8. $(-1, -1, 0)$

9. 6 ; $\langle \frac{1}{3}, -\frac{2}{3}, -\frac{2}{3} \rangle$

10. 10

Types: Number, Function of *variable*(s), 2D vector of numbers, 3D vector of numbers, 2D vector of functions, 3D vector of functions, equation of a plane, parametric equation of a line, equation of a line, equation of a surface, equation of a line, DNE (does not exist).

Sign the following declaration:

I Hereby declare that all the work was done by myself. I was allowed to use Maple, calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 90 minutes on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed:

1. (10 pts.) Find a parametric representation, in terms of the parameter, t, of the line of intersection of the planes $y = 0$ and $z = 0$. What is the usual name for that line?

The **type** of the answers is: parametric equation of a line in 3D space

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ans. $(t, 0, 0)$, $-\infty < t < \infty$; The x-axis

The **short way** (perfectly OK!) is to say that obviously a typical point on the line of intersection of $y = 0$ and $z = 0$ looks like $(x, 0, 0)$ for some x, but the traditional letter for a parameter is t so it is the set of all points $(t, 0, 0)$ where t goes from $-\infty$ to ∞
The **long way** is to note that the vector normal to $y = 0$ is **j** and the vector normal to $z = 0$ is **k** so the line of intersection is in the direction $\mathbf{j} x \mathbf{k} = \mathbf{i}$, so it is of the form $point + t(1, 1, 0)$ where *point* is any point common to both planes, the simplest being $(0, 0, 0)$.

2. (10 points) Use any method to find $\frac{\partial^2 h}{\partial r^2}$ at $(q, r) = (1, 1)$ where $h(u, v) = u^5\, v^6$, $u = q^2\, r$, $v = q\, r^2$

The **type** of the answer is: number

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ans. 272

I told you *use any method*. Superficially it looks like that you need the chain rule, and some people did it that way, and got the right answer and full credit (but it took them a long time!). Other people messed up.
In this simple problem, it is much more efficient to express the function **directly** in terms of q and r and then it is a piece of cake.
The function is $u^5\, v^6 = (q^2 r)^5 (q r^2)^6 = q^{10} r^5 \cdot q^6 r^{12} = q^{16} r^{17}$
So $h_r = 17 q^{16} r^{16}$ and $h_{rr} = 17 \cdot 16 \cdot q^{15} r^{16} = 272$
Finally, don't forget!, plug-in $q = 1$ $r = 1$, and the type of the final answer is **number**.

3. (10 points) Find the point on the line $y = x + 6$ (in the xy-plane) where the rate of change of the function $f(x, y) = x^2 + y^2$ in the direction $< 1, 1 >$ is zero.

The **type** of the answer is: point

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ans. $(-3, 3)$

The **gradient** of f, $< f_x, f_y >$ equals $< 2x, 2y >$. The directional derivative in direction $< 1, 1 >$ is $< 2x, 2y > . < 1, 1 > /\sqrt{2} = \sqrt{2}(x + y)$. Setting this equal to 0 we get that $x + y = 0$. But out point also lies on the line $y = x + 6$. Solving the simple system of two equations and two unknowns x, y:

$$\{x + y = 0, y = x + 6\} \quad ,$$

we get $-x = x + 6$ so $-2x = 6$ so $x = -3$ and $y = -(-3) = 3$. So the desired point is $(-3, 3)$.

4. (10 points) Find an equation of the tangent plane to the following surface at the given point

$$z = x^2 + y^3 \quad , \quad (1,2,9) \quad .$$

The **type** of the answer is: Equation of a plane

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ans. z= 2x+12y-17

First way: Use the fact that for a surface given **explicitly** in the format $z = f(x,y)$ the equation of the tangent plane through (x_0, y_0, z_0) (provided of course the point is a good one, i.e. $f(x_0, y_0) = z_0$) is

$$z - z_0 = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - x_0) \quad .$$

Step 0: Verify that $f(1,2) = 1^2 + 2^3 = 9$ (yea!)

Now

$$f_x = 2x \quad , \quad f_y = 3y^2$$

So

$$f_x(1,2) = 2 \quad , \quad f_y(1,2) = 3 \cdot 2^2 = 12$$

So we get

$$z - 9 = 2(x-1) + 12(y-2)$$

Simplifying

$$z - 9 = 2x - 2 + 12y - 24$$

Simplifyiny more

$$z = 2x + 12y - 17$$

Second way: Convert it to **implicit format** $z - x^2 - y^3 = 0$ and use the formula $grad(f)(x_0, y_0, z_0). < x - x_0, y - y_0, z - z_0 >= 0$.

5. (10 points) Compute $f_{xyy}(1,1)$ (aka $\frac{\partial^3 f}{\partial x \partial^2 y}(1,1)$), if $f(x,y) = \ln(x^2 + y)$.

The **type** of the answer is: Number

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ans. $\frac{1}{2}$.

$f_x = \frac{2x}{x^2+y}$ (by the chain rule of calc1). This can be written $f_x = 2x \cdot (x^2+y)^{-1}$
Moving right along $f_{xy} = 2x \cdot (-1)(x^2+y)^{-2}$ and $f_{xyy} = 2x \cdot (-1)(-2)(x^2+y)^{-3}$, so

$$f_{xyy}(x,y) = \frac{4x}{(x^2+y)^3} \quad .$$

Finally (don't forget!) $f_{xyy}(1,1) = \frac{4 \cdot 1}{(1^2+1)^3} = \frac{4}{8} = \frac{1}{2}$.

6. (10 points) Use Linearization to approximate $\sqrt{(3.001)^2 + (3.999)^2}$. Part of the challenge is to find the appropriate function $f(x, y)$ and the appropriate 'nice' point (x_0, y_0).

The **type** of the answer is: Number

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ans. $\frac{24999}{5000}$ OR 4.9998 (both are acceptable)

The underlying **function** is

$$f(x, y) = \sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2} \quad .$$

The point of interest is $(3.001, 3.999)$. The **nice** point nearby is $(x_0, y_0) = (3, 4)$.
The **Linearization** of $f(x, y)$ at (x_0, y_0) is

$$L(x, y) = f_x(x_0, y_0)\,(x - x_0) \;+\; f_y(x_0, y_0)\,(y - y_0) \quad .$$

$f_x = \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2x = \frac{x}{x^2+y^2} \quad .$
$f_y = \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2y = \frac{y}{x^2+y^2} \quad .$
So $f_x(3, 4) = \frac{3}{\sqrt{3^2+4^2}} = \frac{3}{5}$, $f_y(3, 4) = \frac{4}{\sqrt{3^2+4^2}} = \frac{4}{5}$.
Getting

$$L(x, y) = 5 + \frac{3}{5}\,(x - 3) \;+\; \frac{4}{5}\,(y - 4) \quad .$$

Warning: do **not** simplify. Leave it like that! So

$$L(3.001, 3.999) = 5 + \frac{3}{5}\,(3.001 - 3) \;+\; \frac{4}{5}\,(3.999 - 4) \;=\; 5 + \frac{3}{5}\,(0.001) \;+\; \frac{4}{5}\,(-0.001)$$

$$= 5 - \frac{1/1000}{5} = \frac{24999}{5000} = 4.9998 \quad .$$

Remark: The exact value (up to 10 decimal points) is 4.999800196 (not bad!)

7. (10 points) Decide whether the following limit exists. If it does find it. If it does not **Explain** why not?

$$\lim_{(x,y,z)\to(1,1,1)} \frac{x+2y+3z-6}{3x+2y+z-6}$$

Hint: To prove that a limit **exists** at a designated point you first try to plug it in. If there are no issues you are done, and the value is the limit. Failing this, you try to use **algebra** to simplify and then plug it in, if possible. Failing this you try to prove that it does **not** exist. One way of doing it is to pick any two **different** lines through the point in question and show that if you approach the designated point via these lines you get **different** limits (in the sense of single-variable calculus).

Remark: This was the most challenging problem, and most people did not get it. Please study it carefully.
Plugging in gives 0/0. Trying to simplify also fails. So we are suspecting that the limit does not exist.
We can do some preliminary investigations plugging into the function

$$f(x,y,z) = \frac{x+2y+3z-6}{3x+2y+z-6}$$

Random values **near** the **designated point** $(1,1,1)$. For example

$$f(1,1,1.0001) = 3 \quad , \quad f(1,1.0001,1) = 1$$

1 and 3 are very far apart! So this is an **indication** that the limit **probably** does not exist.
In order to prove it **conclusively** we need to find any two **different** lines **passing through** the designated point $(1,1,1)$ and plug-it in and take the limits in the sense of calc1 (that DO EXIST!, some people claimed that the calc 1 limits do not exist!).

A convenient line is $(1+t,1,1)$ $(-\infty < t < \infty)$, getting that on **this line**

$$f(1+t,1,1) = \frac{(1+t)+2+3-6}{3(1+t)+2+1-6} = \frac{t}{3(1+t)+2+1-6} = \frac{t}{3t} = \frac{1}{3}$$

So along this line the **value** of the function t is **always** $\frac{1}{3}$, and obviously the **limit** in the sense of calc1 **exists** and equals $\frac{1}{3}$.
Another convenient line is $(1,1+t,1)$ $(-\infty < t < \infty)$, getting that on **this line**

$$f(1,1+t,1) = \frac{1+2(1+t)+3-6}{3+2(1+t)+1-6} = \frac{2t}{2t} = \frac{t}{t} = 1.$$

So along this line the **value** of the function t is **always** 1, and obviously the **limit** in the sense of calc1 **exists** and equals 1.

But since these two calc1 limits do **not** agree, this is proof that the limit (in the sense of calc3) **does not exist**.

Comments:

1. You can take **any** two lines . Of course it is good to take convenient line. But the lines **must pass through the designated point**, **not** the origin (unless the point in question happened to be, as often is the case in the homework, and this confused many students) is the origin $(0, 0, 0$.

2. Another, more general approach is to consider an 'abstract' (general) line **passing through** $(1, 1, 1)$

$$(1, 1 + at, 1 + bt)$$

For a and b arbitrary numbers, find the limit along that line as an **expression** in a, b, and since it **depends** on a, b the limit does not exist.

3. Yet another (legal!) approach , for all those who **love** the origin, is to **convert** the problem to a limit about the origin doing a change of variable

$$\bar{x} = x - 1 \quad , \quad \bar{y} = y - 1 \quad , \quad \bar{z} = z - 1 \quad ,$$

Getting that we have to study

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x + 2y + 3z}{3x + 2y + z}$$

and then take lines through the origin! (Here we **renamed** $\bar{x}$ by x , $\bar{y}$ by y , and $\bar{z}$ by z).

Warning: I was disappointed that quite a few people 'faked' it (unintenionaly) writing **gibberish**. If you are not sure, please exercise intellectual honesty and admit that you don't know how to do it.

8. (10 points) Find the minimum **location(s)** (i.e the **point(s)** where the function is the smallest) of the function

$$f(x, y, z) = x + 2y - z$$

in the **closed pyramid** whose vertices are

$$(1, 1, 0) \quad , \quad (-1, 1, 0), \quad (1, -1, 0), \quad (-1, -1, 0), \quad (0, 0, 1)$$

The **type** of the answer(s) is: point

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ans. $(-1, -1, 0)$

The function is **linear** so there are no critical points (not even along the edges). All we have to do is plug-in the function at the five vertices and see who gives the minimal value.

$$f(1, 1, 0) = 3 \quad , \quad f(-1, 1, 0) = 1 \quad , \quad f(1, -1, 0) = -1 \quad , \quad f(-1, -1, 0) = -3 \quad , \quad f(0, 0, 1) = -1$$

The minimum value is -3 at it **happens** at the location $(-1, -1, 0)$.

9. (10 points) A certain particle of mass 2 kilograms has position function, expressed in meters, where t is time, expressed in seconds,

$$\mathbf{r}(t) = \langle\, e^t, 2\cos t, 2\cos t \,\rangle \quad ,$$

(i) Find the **magnitude** of the force acting on it at time $t = 0$. **Explain!**

(ii) Find the **unit direction** of that force.

Reminder from physics: $F = ma$, i.e. the force equals the mass times the acceleration.

The **types** of the answers are (i) non-negative number (it is OK to say number)

(ii) vector (or even better unit vector)

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ans. (i) magnitude : 6 Newtons ; (ii) unit direction:$\langle \frac{1}{3}, -\frac{2}{3}, -\frac{2}{3} \rangle$

$$\mathbf{r}'(t) = \langle\, e^t, -2\sin t, -2\sin t \,\rangle \quad ,$$
$$\mathbf{r}''(t) = \langle\, e^t, -2\cos t, -2\cos t \,\rangle \quad ,$$

$$\mathbf{r}''(0) = \langle\, 1, -2, -2 \,\rangle \quad ,$$
$$\mathbf{F} \;=\; 2\langle\, 1, -2, -2 \,\rangle \;=\; \langle\, 2, -4, -4 \,\rangle \quad ,$$
$$|\mathbf{F}| \;=\; |\langle\, 2, -4, -4 \,\rangle| \;=\; \sqrt{2^2 + (-4)^2 + (-4)^2} = \sqrt{36} = 6 \quad .$$

direction: $\langle\, 2, -4, -4 \,\rangle / 6 = \langle\, \frac{1}{3}, -\frac{2}{3}, -\frac{2}{3} \rangle$

10. (10 points) A certain function $f(x, y, z)$ depends on three variables called x, y and z. At a certain time the rate of change of this function with respect to time happens to be 140. It is also known, that at that same time

• The rate of change of $f(x, y, z)$ with respect to x is 1

• The rate of change of $f(x, y, z)$ with respect to y is 2

• The rate of change of $f(x, y, z)$ with respect to z is 3

• The rate of change of y with respect to time is **two times** The rate of change of x with respect to time.

• The rate of change of z with respect to time is **three times** The rate of change of x with respect to time.

What is the rate of change of x with respect to time at that time?

The **type** of the answer is: Number

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ans. 10

We use the **Chain rule** in its *abstract* form

$$\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} \cdot \frac{dy}{dt} + \frac{df}{dz} \cdot \frac{dz}{dt} \quad .$$

Incorporating this **data**

$$140 = 1 \cdot \frac{dx}{dt} + 2 \cdot \frac{dy}{dt} + 3 \cdot \frac{dz}{dt} \quad .$$

But

$$\frac{dy}{dt} = 2\frac{dx}{dt} \quad , \quad \frac{dz}{dt} = 3\frac{dx}{dt} \quad ,$$

So going back

$$140 = 1 \cdot \frac{dx}{dt} + 2 \cdot (2\frac{dx}{dt}) + 3 \cdot (3\frac{dx}{dt}) \quad ,$$

giving

$$140 = \frac{dx}{dt}(1 + 4 + 9) = \frac{dx}{dt} \cdot 14$$

Giving

$$\frac{dx}{dt} = 10 \quad .$$

Going back to English: The rate of change of x with respect to time, at that above instant, is 10.