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SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as `finalFirstLast.pdf` to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

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1. -18

2. $\int_0^{\frac{1}{2}} \int_{\frac{1}{4}}^1 f(x,y)\,dx\,dy \,+\, \int_{\frac{1}{2}}^1 \int_{y^2}^1 f(x,y)\,dx\,dy$

3. $z = -\frac{1}{2}x - \frac{5}{6}y + \frac{7}{18}\pi$

4. $3\mathbf{i} - 6\mathbf{j} + 9\mathbf{k}$ OR $\langle 3, -6, 9\rangle$

5. $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$

6. $-4\sqrt{3}$

7. $6\cos 2$ OR -2.496881019

8. 8π

9. -15

10. $(\frac{3}{4}, -1)$, saddle point.

11. $3\frac{1}{3000}$ OR $\frac{9001}{3000}$ OR $3.0003333\ldots$.

12. $\frac{\sqrt{2}}{6}$

13. $\int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{\pi} \int_0^2 \rho^6 \sin^4\phi\, \sin^2\theta\, \cos^2\theta\, d\rho d\theta d\phi$

14. $\frac{1}{3}$

15. $\int_0^1 \int_u^1 2\sqrt{u^4 + 4u^2v^2 + v^4}\, dv\, du$ OR $\int_0^1 \int_0^v 2\sqrt{u^4 + 4u^2v^2 + v^4}\, du\, dv$

16. 14

17. 0

Sign the following declaration:

I Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed:

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \quad .$$

1. (12 pts.) **Without using Maple (or any software)** Compute the **vector-field line integral**

$$\int_C (cos\,(e^{\sin x}) + 5y)\,dx \,+\, (sin\,(e^{\cos y}) \,+\, 11x\,)\,dy \quad ,$$

over the path consisting of the five line segments (in that order)

$$(1,0) \to (-1,0) \to (-1,1) \to (0,2) \to (1,1) \to (1,0) \quad .$$

Explain!

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ans. -18

Since this is a **closed** path, we should use **Green's Theorem**. $Q_x - P_y = 11 - 5 = 6$ and we need to integrate it over the region inside the curve. Since the integrand is constant, i.e. 6, the area-integral is simply 6 times the area. This is a "house" where the roof is the triangle with vertices $(-1,1),(1,1),(0,2)$ whose area is $2 \cdot 1/2 = 1$, and the main part of the "house" is the rectangle with vertices $(1,0),(-1,0),(-1,1),(1,1)$ whose area is $2 \cdot 1 =$, so the total area of the region is 3. So Green's theorem tells you that the answer is 18. **But** notice the **orientation** it is **clockwise**, so we have to multiply by -1, getting that the final answer is -18.

2. (12 points) Change the order of integration

$$\int_{\frac{1}{4}}^{1} \int_{0}^{\sqrt{x}} f(x,y)\,dy\,dx \qquad .$$

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ans. $\int_0^{\frac{1}{2}} \int_{\frac{1}{4}}^{1} f(x,y)\,dx\,dy \; + \int_{\frac{1}{2}}^{1} \int_{y^2}^{1} f(x,y)\,dx\,dy +$

The region in Type I format is

$$\{(x,y) : \frac{1}{4} < x < 1, 0 < y < \sqrt{x}\} \quad .$$

Plotting it, we see that this is the region between the vertical lines $x = \frac{1}{4}$ and $x = 1$ and under the curve $y = \sqrt{x}$.
The projection of this region on the y axis is the interval $0 < y < 1$ but now there is no consistent "beginning" to a typical horizontal cross-section. From $y = 0$ to $y = \frac{1}{2}$ it is the vertical line $x = \frac{1}{4}$, but starting at $y = \frac{1}{2}$ it is the curve $y = \sqrt{x}$ which is the same as $x = y^2$. The end is consistent, it is always $x = 1$. So in the Type II description we need to break it up into two parts.

$$\{(x,y) : 0 < y < \frac{1}{2}, \frac{1}{4} < x < 1\} \cup \{(x,y) : \frac{1}{2} < y < 1, y^2 < x < 1\} \quad .$$

This explains the answer.

3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) \;=\; 7 \qquad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

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ans. $z = -\frac{1}{2} - \frac{5}{6}y + \frac{7}{18}\pi \quad .$

First we make sure that the point lies on the surface, by plugging-it-in. It does!
A **normal vector** at the point is the **gradient** of $2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z)$ that is

$$\langle -2\sin(x+y) - 4\sin(x+z), -2\sin(x+y) - 8\sin(y+z), -4\sin(x+z) - 8\sin(y+z)\rangle \quad .$$

Plugging in $x = \frac{\pi}{6}, y = \frac{\pi}{6}, z = \frac{\pi}{6}$, we get that a normal vector is

$$\langle -2\sin(\frac{\pi}{3}) - 4\sin(\frac{\pi}{3}), -2\sin((\frac{\pi}{3}) - 8\sin((\frac{\pi}{3}), -4\sin((\frac{\pi}{3}) - 8\sin((\frac{\pi}{3})\rangle \quad .$$

$$\frac{\sqrt{3}}{2}\langle 6, -10, -12\rangle \quad .$$

Since we can divide by anything (non-zero), a more user-friendly normal vector is $\langle 3, 5, 6\rangle$.
So an equation of a tangent plane at that point, in **implicit format** is

$$2(x - \frac{\pi}{6}) + 5(y - \frac{\pi}{6}) + 6(z - \frac{\pi}{6}) \;=\; 0 \quad .$$

Rearranging, and solving for z, we get

4. (16 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k} \quad , \quad \mathbf{b} \times \mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad , \quad \mathbf{a} \times \mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad .$$

What is

$$(\mathbf{a} + \mathbf{b} + \mathbf{c}) \times (2\mathbf{a} - \mathbf{b} + 3\mathbf{c}) \qquad ?$$

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ans. $3\mathbf{i} - 6\mathbf{j} + 9\mathbf{k}$ OR $\langle 3, -6, 9 \rangle$

Using the **distributive property** of the **cross-product** (a fancy name for "opening parentheses") we have

$$2\,\mathbf{a} \times \mathbf{a} - \mathbf{a} \times \mathbf{b} + 3\,\mathbf{a} \times \mathbf{c}$$

$$+2\,\mathbf{b} \times \mathbf{a} - \mathbf{b} \times \mathbf{b} + 3\,\mathbf{b} \times \mathbf{c}$$

$$+2\,\mathbf{c} \times \mathbf{a} - \mathbf{c} \times \mathbf{b} + 3\,\mathbf{c} \times \mathbf{c}$$

But the cross-product of a vector with itself is the **zero vector**, so we get rid of $\mathbf{a} \times \mathbf{a}$, $\mathbf{b} \times \mathbf{b}$, $\mathbf{c} \times \mathbf{c}$.
Another important property is that $\mathbf{y} \times \mathbf{x} = -\mathbf{x} \times \mathbf{y}$,
so $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$, $\mathbf{c} \times \mathbf{a} = -\mathbf{a} \times \mathbf{c}$, $\mathbf{c} \times \mathbf{b} = -\mathbf{b} \times \mathbf{c}$.
So we get

$$-\mathbf{a} \times \mathbf{b} + 3\,\mathbf{a} \times \mathbf{c}$$

$$-2\,\mathbf{a} \times \mathbf{b} + 3\,\mathbf{b} \times \mathbf{c}$$

$$-2\,\mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$$

$$= -3\,\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} + 4\,\mathbf{b} \times \mathbf{c} \quad .$$

Finally, using the data, we get

$$-3\,(\mathbf{i} + \mathbf{j} - \mathbf{k}) + 2\,\mathbf{i} + \mathbf{j} + 2\,\mathbf{k} + 4\,(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

$$3\,\mathbf{i} - 6\,\mathbf{j} + 9\,\mathbf{k} \quad .$$

This is the answer. In the usual notation it is $\langle 3, -6, 9 \rangle$.

5. (12 points) Find the three angles of the triangle ABC where

$$A = (0,0,0) \quad , \quad B = (1,0,1) \quad , \quad C = (1,1,0) \quad .$$

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ans. The angle at A is: $\frac{\pi}{3}$ radians ;

The angle at B is: $\frac{\pi}{3}$ radians ;

The angle at C is: $\frac{\pi}{3}$ radians ;

First Way:
The length of AB is $\sqrt{(1-0)^2+(0-0)^2+(1-0)^2} = \sqrt{2}$
The length of AC is $\sqrt{(1-0)^2+(1-0)^2+(0-0)^2} = \sqrt{2}$
The length of BC is $\sqrt{(1-1)^2+(1-0)^2+(0-1)^2} = \sqrt{2}$
Since all the sides are the same it is an **equilateral triangle** so all the angles are the same and equal to $\frac{pi}{3}$.

First Way: Let θ_A be the angle at A, then

$$\cos\theta_A = \frac{AB.AC}{|AB||AC|} \quad ,$$

The vector AB is $\langle 1,0,1 \rangle >$. The vector AC is $\langle 1,1,0 \rangle >$ So

$$\cos\theta_A = \frac{2}{\sqrt{2}\sqrt{2}} = \frac{1}{2} \quad ,$$

and $\theta_A = \frac{\pi}{6}$. Similarly for θ_B and θ_C.

6. (12 points) Find the directional derivative of

$$f(x, y, z) = x^3 + y^3 + z^3 + xyz \quad ,$$

at the point $(1, 1, 1)$ in a direction pointing to the point $(-1, -1, -1)$.

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ans. $-4\sqrt{3}$

$$grad(f) = \langle\, f_x \,,\, f_y \,,\, f_z \,\rangle$$

$$\langle 3x^2 + yz \,,\, 3y^2 + xz \,,\, 3z^2 + xy \rangle$$

Plugging-in $x = 1, y = 1, z = 1$, we get

$$grad(f)(1, 1, 1) = \langle 4, 4, 4 \rangle$$

The **direction** is $\langle -1, -1, -1 \rangle - \langle 1, 1, 1 \rangle = \langle -2, -2, -2 \rangle$. The **unit direction u** is $\langle -1, -1, -1 \rangle / \sqrt{3}$.
Hence the **directional derivative** $grad(f).\mathbf{u}$ is

$$\langle 4, 4, 4 \rangle, \langle -1, -1, -1 \rangle / \sqrt{3} = -3 \cdot 4/\sqrt{3} = -4\sqrt{3} \quad ,$$

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\frac{\partial g}{\partial u}$$

at $(u, v) = (0, 1)$, where

$$g(x, y) = 3x^2 - 3y^2 \quad ,$$

and

$$x = e^u \cos v \quad , \quad y = e^u \sin v \quad .$$

---

ans. 6 cos 2

The chain rule says:

$$g_u = g_x \cdot x_u + g_y \cdot y_u$$

In this problem

$$g_x = 6x \quad , \quad g_y = -6y$$

$$x_u = e^u \cos v \quad , \quad y_u = e^u \sin v$$

At $u = 0$, $v = 1$, $x = \cos 1$, $y = \sin 1$, so

$$g_x = 6 \cos\, 1 \quad , \quad g_y = -6 \sin\, 1$$

$$x_u = \cos\, 1 \quad , \quad y_u = \sin\, 1$$

So at this point

$$g_u = 6 \cos\, 1 \cdot \cos\, 1 - 6 \sin 1 \cdot \sin\, 1 = 6(\cos^2 1 - \sin^2 1) = 6 \cos\, 2 \quad .$$

(Here we used the famous trig identity $\cos^2 \theta - \sin^2 \theta = \cos\, 2\theta$.)

8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle\, 3x + cos(y^3 + yz)\,,\, -2y + e^{x+z^2}\,,\, 5z + \sin(xy^3 + e^x)\,\rangle \qquad ,$$

and S is the closed surface in 3D space bounding the region

$$\{(x, y, z) : x^2 + y^2 + z^2 < 4 \quad and \quad x > 0 \quad and \quad y < 0 \quad and \quad z > 0\} \quad .$$

---

ans. 8π

This calls for the **divergence theorem**.

$$div(\mathbf{F}) = 3 - 2 + 5 = 6 \quad .$$

So the value that we need is the integral of 6 over the region. But taking 6 out, it is 6 times the volume of the region. The region is one-eighth of a sphere radius 2, so the volume is

$$\frac{1}{8}\frac{4}{3} \cdot 2^3 = \frac{4}{3}\pi \quad .$$

Multiplying by 6 gives the answer, 8π.

9. (12 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle 3z\,,\, 2x\,,\, y+z \rangle \quad ,$$

and S is the oriented surface

$$z = 2x + 3y \quad , \quad 0 < x < 1, \quad 0 < y < 1 \quad ,$$

with **upward pointing** normal.

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ans. -15

There are no short cuts here. We need the formula

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \quad .$$

Here

$$P = 3z \quad , \quad Q = 2x \quad , \quad R = y + z$$

$$g(x,y) = 2x + 3y$$

and the region is $D = \{(x,y) : 0 < x < 1, 0 < y < 1\}$.
The **integrand** is

$$(-3z)(2) - (2x)(3) + y + z = -6x + y - 5z \quad .$$

Replacing z by $2x + 3y$ this gives

$$-6x + y - 5(2x + 3y) = -6x + y - 10x - 15y = -16x - 14y \quad .$$

Integrating we get

$$\int_0^1 \int_0^1 (-16x - 14y)\, dxdy = -16 \int_0^1 \int_0^1 x\, dxdy - -14 \int_0^1 \int_0^1 y\, dxdy = -15 \quad .$$

10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x,y) = 4x - y^2 - \ln(2x+y) \quad ,$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

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ans. $(\frac{3}{4}, -1)$, saddle point.

$$f_x = 4 - \frac{2}{2x+y} \quad , \quad f_y = -2y - \frac{1}{2x+y} \quad .$$

For future reference

$$f_{xx} = \frac{4}{(2x+y)^2} \quad , \quad f_{xy} = \frac{2}{(2x+y)^2} \quad , \quad f_{yy} = -2 + \frac{1}{(2x+y)^2} \quad .$$

Solving

$$4 - \frac{2}{2x+y} = 0 \quad , \quad -2y - \frac{1}{2x+y} = 0 \quad .$$

From the first equation $\frac{1}{2x+y} = 2$. Putting it in the second equation we get $-2y - 2 = 0$ so $y = -1$. Going back to the first equation we get $x = \frac{3}{4}$.
So there is only one **critical point** $(\frac{3}{4}, -1)$.
Plugging into f_{xx}, f_{xy}, f_{yy} we get

$$f_{xx} = 16 \quad , \quad f_{xy} = 8 \quad , \quad f_{yy} = 3$$

So the **discriminant** $D = f_{xx} f_{yy} - (f_{xy})^2$ at that point is $16 \cdot 2 - 8^2 = -32$. Since it is **negative** the point is a **saddle point**.

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1,1,2)$, approximate $f(1.001, 0.999, 2.001)$ if

$$f(x,y,z) = \sqrt{2x^2+3y^2+z^2} \quad .$$

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ans. $3\,\frac{1}{3000}$ OR $\frac{9001}{3000}$ OR $3.0003333\ldots$.

$$f = (2x^2+3y^2+z^2)^{\frac{1}{2}}$$

$$f_x = \frac{1}{2}(2x^2+3y^2+z^2)^{-\frac{1}{2}}(4x) = \frac{2x}{\sqrt{2x^2+3y^2+z^2}}$$

$$f_y = \frac{1}{2}(2x^2+3y^2+z^2)^{-\frac{1}{2}}(6y) = \frac{3y}{\sqrt{2x^2+3y^2+z^2}}$$

$$f_z = \frac{1}{2}(2x^2+3y^2+z^2)^{-\frac{1}{2}}(2z) = \frac{z}{\sqrt{2x^2+3y^2+z^2}}$$

At $x=1$, $y=1$, $z=2$ we have

$$grad(f) = \langle \frac{2}{3}, 1, \frac{2}{3} \rangle \quad .$$

The **linearization** is

$$L(x,y,z) = f(1,1,2) + \frac{2}{3}(x-1) + 1\cdot(y-1) + \frac{2}{3}(z-2) == 3 + \frac{2}{3}(x-1) + 1\cdot(y-1) + \frac{2}{3}(z-2) \quad .$$

Plugging in the acual values $x=1.001$, $y=0.999$, $z=2.001$ gives the approximation

$$3 + \frac{2}{3}(\frac{1}{1000}) + 1\cdot(-\frac{1}{1000}) + \frac{2}{3}(\frac{1}{1000}) = 3 + \frac{1}{3000} = \frac{9001}{3000} \quad .$$

12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^x x\,dy\,dx + \int_{\frac{\sqrt{2}}{2}}^1 \int_0^{\sqrt{1-x^2}} x\,dy\,dx \quad .$$

Explain!

---

ans. $\frac{\sqrt{2}}{6}$

The region of integration is

$$D = \{0 < x < \frac{\sqrt{2}}{2}, 0 < y < x\} \cup \{\frac{\sqrt{2}}{2} < x < 1, 0 < y < \sqrt{1-x^2}\} \quad .$$

If you draw a picture, this is exactly the one-eighth of the unit circle

$$\{(r,\theta) : 0 < r < 1 \quad , \quad 0 < \theta < \frac{\pi}{4}\} \quad .$$

Converting to polar coordinates, we get

$$\int_0^{\frac{\pi}{4}} \int_0^1 (r\cos\theta) r\,dr\,dtheta$$

$$\int_0^{\frac{\pi}{4}} \int_0^1 (r^2\cos\theta) r\,dr\,dtheta$$

$$\left(\int_0^1 r^2\,dr\right)\left(\int_0^{\frac{\pi}{4}} \cos\theta\right) = \frac{1}{3}\cdot\sin\frac{\pi}{4} = \frac{\sqrt{2}}{6} \quad .$$

13. (12 points) Convert the triple iterated integral

$$\int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{\pi} \int_0^2 \rho^6 \sin^4\phi \, \sin^2\theta \, \cos^2\theta \, d\rho d\theta d\phi$$

to **spherical coordinates**. **Do not evaluate.**

---

ans. $\int_0^2 \int_{\frac{\pi}{2}}^{\pi} \int_0^{\frac{\pi}{2}} \rho^6 \sin^4\phi \, \sin^2\theta \, \cos^2\theta \, d\rho d\theta d\phi$

The challenging part is figuring out the region in spherical coordinates. Sincs z is positive ϕ goes from 0 to $\frac{\pi}{2}$ (the Northern hemisphere). Since x is negative and y is positive in the integration region we are talking about the **second quadrant** so θ goes from $\frac{\pi}{2}$ to π. Regarding the integrand, you use the 'dictionary'

$$x = \rho \sin\phi \cos\theta \quad , \quad y = \rho \sin\phi \sin\theta \quad , \quad z = \rho \cos\phi \quad , \quad dx\,dy\,dz \;=\; \rho^2 \sin\phi, d\rho\, d\phi\, d\theta \quad .$$

14. (12 points) Find the curvature of the curve

$$\mathbf{r}(t) = \langle 5, 3\sin t, 3\cos t \rangle$$

at the point where $t = \frac{\pi}{3}$.

---

ans. $\frac{1}{3}$

Shortcut way: This is a circle of radius 3 so the curvature is $\frac{1}{3}$ (everywhere, not just at $t = \pi/3$).
Usual way: Find $\mathbf{r}'(t)$, $\mathbf{r}''(t)$. Plug-in $t = \pi/3$, and use the formula for the curvature

$$\frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \quad .$$

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parameterically by

$$\mathbf{r}(u,v) = \langle u^2 \, , \, uv \, , \, v^2 \rangle \quad , \quad 0 < u < v < 1 \quad .$$

---

ans. $\int_0^1 \int_u^1 2\sqrt{u^4 + 4u^2v^2 + v^4}\, dv\, du$

$$\mathbf{r}_u = \langle 2u \, , \, v \, , \, 0 \rangle \quad , \quad 0 < u < v < 1 \quad .$$

$$\mathbf{r}_v = \langle 0 \, , \, u \, , \, 2v \rangle \quad , \quad 0 < u < v < 1 \quad .$$

$$\mathbf{r}_u \times \mathbf{r}_v =$$

$$\langle 2v^2, -4uv, 2u^2 \rangle \quad .$$

So

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{4u^4 + 16u^2v^2 + 4u^4} = 2\sqrt{2u^4 + 4u^2v^2 + u^4} =$$

Now we integrate over the region $\{(u,v) : 0 < u < 1, 0 < u < v\}$.
Comment: People who wrote $\int_0^1 \int_0^v 2\sqrt{u^4 + 4u^2v^2 + v^4}\, du\, dv$ also got full credit, if you decide to make v the 'boss'.

16. (12 points) Let

$$f(x, y, z) \;=\; xy^2z^3 \quad ,$$

and let

$$g(x, y, z) \;=\; x + y^2 + z^3 \quad .$$

compute the dot-product

$$grad(f) \,.\, grad(g) \qquad .$$

at the point $(1, 1, 1)$.

---

ans. 14

$$grad(f) = \langle y^2z^2, 2xyz^3, 3xy^2z^2 \rangle$$

$$grad(g) = \langle 1, 2y, 3z^2 \rangle$$

$$grad(f)(1, 1, 1) = \langle 1, 2, 3 \rangle$$

$$grad(g)(1, 1, 1) = \langle 1, 2, 3 \rangle$$

So the dot-product is $1^2 + 2^2 + 3^2 = 14$.

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w} \quad .$$

---

ans. 0

When you plug-it-in you get 0/0, but after you simplify you get, using $(a^2-b^2)/(a-b) = a+b$, that the function equals $x+y+z+w$. Now you plug-it-in and get 0.