**Solution to the "QUIZ" for Lecture 24**

By using Stokes' Theorem, or otherwise, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$F(x,y,z) = (yz + 2y + 3z)\mathbf{i} + (xz + 2x + 4z)\mathbf{j} + (xy + 3x + 4y)\mathbf{k} \quad ,$$

where $C$ is the curve of intersection of the plane $x+y+z=1$ and the cylinder $x^2+y^2=1$, oriented counterclockwise as viewed from above. Be sure to explain everything.

**Sol.** Stokes's theorem says

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_S curl(\mathbf{F}) \cdot d\,\mathbf{S} \quad ,$$

where $S$ is *any* surface that bounds $C$. This would be very hard to compute *unless* we are lucky and $curl\mathbf{F}$ would happen to be the zero vector $\langle\, 0,0,0 \,\rangle$. In that case we would be done, since if the vector field that is being integrated is the **zero** vector-field, and answer is auomatically 0!

So let's compute $curl(\mathbf{F})$:

$$curl(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz+2y+3z & xz+2x+4z & xy+3x+4y \end{vmatrix}$$

$$\mathbf{i}\left(\frac{\partial}{\partial y}(xy+3x+4y) - \frac{\partial}{\partial z}(xz+2x+4z)\right)$$

$$-\mathbf{j}\left(\frac{\partial}{\partial x}(xy+3x+4y) - \frac{\partial}{\partial z}(yz+2y+3z)\right)$$

$$+\mathbf{k}\left(\frac{\partial}{\partial x}(xz+2x+4z) - \frac{\partial}{\partial y}(yz+2y+3z)\right)$$

$$\mathbf{i}\,(x+4-(x+4)) - \mathbf{j}\,(y+3-(y+3)) + \mathbf{k}\,(z+2-(z+2)) = \mathbf{i}0 - \mathbf{j}0 + \mathbf{k}0 = \langle 0,0,0 \rangle \quad .$$

So we are lucky and the curl is indeed the zero-vector and we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_S curl(\mathbf{F}) \cdot d\,\mathbf{S} = \int\int_S \langle 0,0,0 \rangle \cdot d\,\mathbf{S} = 0 \quad .$$

**Ans.:** 0.

Comments:

1. About %60 of the people got it right.

2. Some people did the "otherwise" method, but didn't finish it. No wonder, doing it directly is much more complicated.

3. Some people said that the answer is 0 because the surface $S$ is **closed**. This is **wrong!**. The *surface* $S$ (whatever it is) is not a closed surface. The **curve** $C$ is closed, but the surface is open of course. If a surface is closed, than it has **no boundary**, and in this problem we are told that the boundary of our surface is $C$. You can never use the argument $S$ is closed when you are asked to find a *line-integral*, $\int_C \mathbf{F} \cdot \mathbf{d}\,\mathbf{s}$, because $S$ is **never** closed (for that scenario). Your only hope at an easy way out is to compute $curl\mathbf{F}$ and see that it is the zero-vector $\langle 0, 0, 0 \rangle$ . The "closed surface" argument works when you have to computer a **surface integral** $\int\int_S curl(\mathbf{F}) \cdot d\mathbf{S}$, where $S$ is the *whole* sphere or the *whole* ellipsoid or whatever, then it is automatically zero.