Solution to the "QUIZ" for Lecture 24

By using Stokes' Theorem, or otherwise, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where

$$F(x,y,z) = (yz+2y+3z)\mathbf{i} + (xz+2x+4z)\mathbf{j} + (xy+3x+4y)\mathbf{k} \quad ,$$

where C is the curve of intersection of the plane $x+y+z=1$ and the cylinder $x^2+y^2=1$, oriented counterclockwise as viewed from above. Be sure to explain everything.

Sol. Stokes's theorem says

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int\int_S curl(\mathbf{F})\cdot d\,\mathbf{S} \quad ,$$

where S is *any* surface that bounds C. This would be very hard to compute *unless* we are lucky and $curl\mathbf{F}$ would happen to be the zero vector $\langle\,0,0,0\,\rangle$. In that case we would be done, since if the vector field that is being integrated is the **zero** vector-field, and answer is auomatically 0!

So let's compute $curl(\mathbf{F})$:

$$curl(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz+2y+3z & xz+2x+4z & xy+3x+4y \end{vmatrix}$$

$$\mathbf{i}\left(\frac{\partial}{\partial y}(xy+3x+4y) - \frac{\partial}{\partial z}(xz+2x+4z)\right)$$

$$-\mathbf{j}\left(\frac{\partial}{\partial x}(xy+3x+4y) - \frac{\partial}{\partial z}(yz+2y+3z)\right)$$

$$+\mathbf{k}\left(\frac{\partial}{\partial x}(xz+2x+4z) - \frac{\partial}{\partial y}(yz+2y+3z)\right)$$

$$\mathbf{i}\,(x+4-(x+4)) - \mathbf{j}\,(y+3-(y+3)) + \mathbf{k}\,(z+2-(z+2)) = \mathbf{i}0 - \mathbf{j}0 + \mathbf{k}0 = \langle 0,0,0\rangle \quad .$$

So we are lucky and the curl is indeed the zero-vector and we have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int\int_S curl(\mathbf{F})\cdot d\,\mathbf{S} = \int\int_S \langle 0,0,0\rangle\cdot d\,\mathbf{S} = 0 \quad .$$

Ans.: 0.

Comment Some people said that the answer is 0 because the surface S is **closed**. This is **wrong!**. The *surface* S (whatever it is) is not a closed surface. The **curve** C is closed, but the surface is open of course. If a surface is closed, than it has **no boundary**, and in this problem we are told that the boundary of our surface is C. You can never use the argument S is closed when you are asked to find a *line-integral*, $\int_C \mathbf{F}\cdot\mathbf{d\,s}$, because S is **never** closed (for that scenario). Your only hope at an easy way out is to compute $curl\mathbf{F}$ and see that it is the zero-vector $\langle 0,0,0\rangle$. The "closed surface" argument works when you have to computer a **surface integral** $\int\int_S curl(\mathbf{F})\cdot d\mathbf{S}$, where S is the *whole* sphere or the *whole* ellipsoid or whatever, then it is automatically zero.