Solutions to MATH 251 (4,6,7), Dr. Z. , Exam 2, Tue., Nov. 21, 2017, SEC 118

FRAME YOUR FINAL ANSWER(S) TO EACH PROBLEM
Do not write below this line

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1. (out of 10)

2. (out of 10)

3. (out of 10)

4. (out of 10)

5. (out of 10)

6. (out of 10)

7. (out of 10)

8. (out of 10)

9. (out of 10)

10. (out of 10)

MAKE SURE TO PUT THE TYPE!
Types: Number, Function of *variable*(s), 2D vector of numbers, 3D vector of numbers, 2D vector of functions (aka 2D vector-field), 3D vector of functions (aka 3D vector field), equation of a plane, parametric equation of a line, equation of a line, equation of a surface, equation of a line, DNE (does not exist), abstract double-integral, abstract triple-integral.

1. (10 pts.)
Find the Jacobian of the transformation from (u, v, w)-space to (x, y, z)-space.

$$x = uv + w \quad , \quad y = uw + v \quad , \quad z = vw + u \quad ,$$

at the point $(u, v, w) = (2, 2, 2)$.

The **type** of the answers is: Number

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ans. -5

The Jacobian is

$$\det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} = \det \begin{pmatrix} v & u & 1 \\ w & 1 & u \\ 1 & w & v \end{pmatrix}$$

Plugging in $u = 2, v = 2, w = 2$, we get

$$\det \begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

$$= 2 \cdot \det \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} - 2 \cdot \det \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} + 1 \cdot \det \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$2 \cdot (1 \cdot 2 - 2 \cdot 2) - 2 \cdot (2 \cdot 2 - 1 \cdot 2) + 1 \cdot (2 \cdot 2 - 1 \cdot 1)$$

$$= 2(-2) - 2(2) + 3 = -4 - 4 + 3 = -5 \quad .$$

2. (10 points altogether)
(i) (3 points) Show that

$$\mathbf{F} = \langle 3x^2yz+yz+\cos(x+y+z)\ ,\ x^3z+xz+\cos(x+y+z)\ ,\ x^3y+xy+\cos(x+y+z)\rangle \quad ,$$

is a conservative vector field.

(ii) (4 point) Find a function $f(x,y,z)$ such that $\mathbf{F} = \bigtriangledown f$.

(iii) (3 points) Find the line-integral $\int_C \mathbf{F}.d\mathbf{r}$ where C is the curve

$$\mathbf{r} = \langle \sin t, \cos t + 1, \sin 2t\rangle \quad , \quad 0 \le t \le \pi \quad .$$

The **types** of the answer is: For (ii) multivariable function For (iii) Number

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answers (ii) $\quad f(x,y,z) = x^3yz + xyz + \sin(x+y+z)$

(iii) $-\sin 2 \quad .$

Sol. of (i): We have to show that the **curl** of the vector-field $\mathbf{F}$, $\bigtriangledown \times \mathbf{F}$ equals the **zero-vector**, $\mathbf{0}$. In other words, $\bigtriangledown \times \mathbf{F} = \langle 0,0,0\rangle$.

$$\bigtriangledown \times \mathbf{F}$$

$$= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2yz + yz + \cos(x+y+z) & x^3z + xz + \cos(x+y+z) & x^3y + xy + \cos(x+y+z) \end{pmatrix}$$

$$\mathbf{i} \cdot \left(\frac{\partial}{\partial y}(x^3y + xy + \cos(x+y+z)) - \frac{\partial}{\partial z}(x^3z + xz + \cos(x+y+z)) \right)$$

$$-\mathbf{j} \cdot \left(\frac{\partial}{\partial x}(x^3y + xy + \cos(x+y+z)) - \frac{\partial}{\partial z}(3x^2yz + yz + \cos(x+y+z)) \right)$$

$$\mathbf{k} \cdot \left(\frac{\partial}{\partial x}(x^3z + xz + \cos(x+y+z)) - \frac{\partial}{\partial y}(3x^2yz + yz + \cos(x+y+z)) \right)$$

$$= \mathbf{i} \cdot ((x^3 + x + \sin(x+y+z) - (x^3 + x + \sin(x+y+z))$$

$$-\mathbf{j} \cdot ((3x^2y + y + \sin(x+y+z)) - (3x^2y + y + \sin(x+y+z))$$

$$+\mathbf{k} \cdot ((3x^2z + z + \sin(x+y+z)) - (3x^2z + z + \sin(x+y+z))$$

$$= \mathbf{i}0 + \mathbf{j}0 + \mathbf{k}0 \; = \; \langle 0,0,0\rangle \quad .$$

Hence $\mathbf{F}$ is a **conservative vector field**.

Sol. to (ii): We need to find a "magic function" $f(x,y,z)$ such that $\bigtriangledown f = \mathbf{F}$. In other words

$$f_x = 3x^2yz+yz+\cos(x+y+z) \quad , \quad f_y = x^3z+xz+\cos(x+y+z) \quad , \quad f_z = x^3y+xy+\cos(x+y+z) \quad .$$

From the first equation (integrating w.r.t. to x) we get

$$f = \int (3x^2yz + yz + \cos(x+y+z))\, dx \; = \; x^3yz + xyz + \sin(x+y+z) + \Phi(y,z) \quad ,$$

where $\Phi(y,z)$ is some function of **only** y and z, yet to be determined.
Since $f_y = x^3z + xz + \cos(x+y+z)$, we have

$$x^3z + xz + \cos(x+y+z) + \Phi_y(y,z) = x^3z + xz + \cos(x+y+z) \quad ,$$

Hence $\Phi_y(y,z) = 0$ and Φ does not depend on y. Hence $\Phi(y,z) = \Psi(z)$. Where $\Psi(z)$ is yet to be determined.
Now we can write:

$$f = x^3yz + xyz + \sin(x+y+z) + \Psi(z) \quad ,$$

Since $f_z = x^3y + xy + \cos(x+y+z)$, we have

$$x^3y + xy + \cos(x+y+z) + \Psi'(z) = x^3y + xy + \cos(x+y+z) \quad ,$$

Hence $\Psi'(z)$, in other words $\Psi(z) = C$, and we can take $C = 0$. Hence we get

$$f(x,y,z) = x^3yz + xyz + \sin(x+y+z) \quad .$$

Sol. to (iii): By the **fundamental theorem of line integrals**, the answer is $f(EndPoint) - f(StartingPoint)$. The starting point, $\mathbf{r}(0) = (0,2,0)$. The end-point is $\mathbf{r}(\pi) = (0,0,0)$, hence

$$\int_C \mathbf{F}.d\mathbf{r} \; = \; f(0,0,0) - f(0,2,0) = (0^3 \cdot 0 \cdot 0 + 0 \cdot 0 \cdot 0 + \sin(0+0+0)) - (0^3 \cdot 2 \cdot 0 + 0 \cdot 3 \cdot 0 + \sin(0+2+0)) = -\sin 2 \quad .$$

Comment: Some people tried to do it directly. This is correct, in principle, but way **too complicated**, and then they complained that they "ran out of time". If you know what you are doing, and do not go on the wrong track, the exam did not take too long, and indeed, quite a few people got almost perfect score.
Use your **common sense**. In an exam, things should not get too complicated, and if things get our of hand, it probably means that you are in the wrong track, and you should move on to another problem.

3. (10 points)
Sketch the region of integration and change the order of integration.

$$\int_1^2 \int_0^{e^x+1} F(x,y)\, dy\, dx$$

The **type** of the answer is: Sum of two Abstract double-integrals

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ans. [Typo corrected Nov. 19, 2020, thank to Zixin Qu who won a dollar]

$$\int_0^{e+1} \int_1^2 F(x,y)\, dx\, dy + \int_{e+1}^{e^2+1} \int_{\ln(y-1)}^2 F(x,y)\, dx\, dy \quad .$$

The Type I description of the region of integration is

$$\{(x,y) \,|\, 1 \leq x \leq 2 \quad , \quad 0 \leq y \leq e^x+1\} \quad .$$

We need to find the Type II description of the same region.

From the point of view of the x-axis as the main road, it starts at $x=1$ and ends at $x=2$, and each vertical cross-section extends from $y=0$ to $y=e^x+1$. So this is the region where

- The floor is the portion of the x-axis from $(1,0)$ to $(2,0)$
- the left wall is the line segment from $(1,0)$ to $(1,e+1)$
- the right wall is the line segment from $(2,0)$ to $(2,e^2+1)$
- the 'roof" is the curve $y=e^x+1$.

Note that $y=e^x+1$ can also be written, from the view-point of y, as $x=\ln(y-1)$.

The projection of this region onto the y-axis is from $y=0$ to $y=e^2+1$, but it is not a simple description. From $y=1$ to $y=e+1$ the left-wall is the vertical line $x=1$, but from $y=e+1$ to $y=e^2+1$ it is the curve $y=e^x+1$, alias $x=\ln(y-1)$, hence the type II description is

$$\{(x,y) \,|\, 1 \leq y \leq e{+}1 \quad , \quad 1 \leq x \leq 2\} \cup \{(x,y) \,|\, e{+}1 \leq y \leq e^2{+}1 \quad , \quad \ln(y{-}1) \leq x \leq 2\} \quad .$$

Hence we get that the original double-integral equals

$$\int_0^{e+1} \int_1^2 F(x,y)\, dx\, dy + \int_{e+1}^{e^2+1} \int_{\ln(y-1)}^2 F(x,y)\, dx\, dy \quad .$$

4. (10 points) Use Largange multipliers (no credit for other methods) to find the smallest value that $x + y + z$ can be, given that $xyz = 1$.

The **type** of the answer is: Number

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ans. 3

We have

We use **Lagrange Multiplies** with $f = x + y + z$, $g = xyz - 1$.

$$\bigtriangledown f \; = \; \langle f_x, f_y, f_x \rangle \; = \; \langle 1, 1, 1 \rangle \quad .$$

$$\bigtriangledown g \; = \; \langle g_x, g_y, g_z \rangle \; = \; \langle yz, xz, xy \rangle \quad .$$

We need to find (x, y, z) and λ such that

$$\bigtriangledown f \; = \; \lambda \bigtriangledown g \quad .$$

Hence

$$\langle 1, 1, 1 \rangle = \lambda \langle yz, xz, xy \rangle \quad .$$

Spelling it out, we have, in addition to the **constraint** $xyz = 1$, the following three equations

$$1 = \lambda yz \quad , \quad 1 = \lambda xz \quad , \quad 1 = \lambda xy \quad .$$

Multiplying the first equation by x, the second equation by y, and the third equation by z:

$$x = \lambda xyz \quad , \quad y = \lambda xyz \quad , \quad z = \lambda xyz \quad .$$

But $xyz = 1$, hence $x = \lambda$, $y = \lambda$, $z = \lambda$. Plugging-into the constraint, we get

$$\lambda^3 \; = \; 1 \quad ,$$

giving $\lambda = 1$.
Hence $x = 1, y = 1, z = 1$. This is a **location**. Plugging-in into the **goal function**, $f(x, y, z) = x + y + z$, we get that the minimum value is $1 + 1 + 1 = 3$.

5. (10 points) Compute the volume integral

$$\int\int\int_E 48\,x\,y\,z\,dV$$

where E is the region in 3D

$$\{(x,y,z)\,|\,0 \le x \le y \le z \le 1\} \quad .$$

The **type** of the answer is: Number

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ans. 1 .

Our region (in 3D) can be written as

$$\{(x,y,z)\,|\,0 \le z \le 1\,,\,0 \le y \le z\;0 \le x \le y\} \quad .$$

Hence the **iterated integral is**

$$\int_0^1 \int_0^z \int_0^y (48xyz)\,dx\,dy\,dz \quad .$$

The **inner integral** is

$$\int_0^y (48xyz)\,dx \;=\; 48yz\int_0^y x\,dx = 48yz\left(\frac{x^2}{2}\Big|_{x=0}^{x=y}\right) \;=\; 48yz(y^2-0^2)/2 \;=\; 24y^3z \quad .$$

The **middle integral** is

$$\int_0^z (24y^3z)\,dy \;=\; 24z\left(\frac{y^4}{4}\Big|_{y=0}^{y=z}\right) = 24z(z^4-0^4)/4 = 6z^5 \quad .$$

The **outer integral** is

$$\int_0^1 6z^5\,dz = 6\frac{z^6}{6}\Big|_{z=0}^{z=1} = z^6\Big|_{z=0}^{z=1} = (1^6-0^6) = 1 \quad .$$

Hence the final answer is the number 1.

Comment: Many people solved (correctly!) a **different** problem. They assumed that the region was $\{(x,y,z)|0 \le x,y,z \le 1\}$, whose explicit description is $\{(x,y,z)\,|\,0 \le x \le 1\,,\,0 \le y \le 1\;0 \le z \le 1\}$. They got the (correct) answer 6 (for the wrong problem).
If you are really advanced, you could have argued that since the **integrand** is symmetric with respect to the variables x,y,z, and the region $\{(x,y,z)\,|\,0 \le x \le 1\,,\,0 \le y \le 1\;0 \le z \le 1\}$ can be broken up to **six** regions according to the orderings of x,y,z, and the integral is the same for each, one can first do the simpler problem, and then divide by 6. But one must be very careful when using such "shortcuts". It is only valid if the integrand is symmetric with respect to x,y,z.

6. (10 points) By converting to polar coordinates, compute

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \frac{\left(x^2+y^2\right)^2}{243\pi}\,dy\,dx$$

The **type** of the answer is: Number

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ans. 1 .

The region of integration is a disc, center origin, and radius 3, whose polar description is

$$\{(r,\theta)\,|\,0 \le r \le 3 \quad , \quad 0 \le \theta \le 2\pi\} \quad .$$

Since $x^2 + y^2 = r^2$, the integrand is $r^4/(243\pi)$ but $dy\,dx$ should be replaced by $r dr\,d\theta$. Hence our integral is

$$\int_0^{2\pi}\int_0^3 \frac{r^4}{243\pi}\, r dr d\theta = \frac{1}{243\pi}\int_0^{2\pi}\int_0^3 r^5\, dr d\theta =$$

$$\frac{1}{243\pi}\left(\int_0^{2\pi} d\theta\right)\left(\int_0^3 r^5\, dr\right)$$

$$= \frac{1}{243\pi}(2\pi)\left(\frac{r^6}{6}\Big|_0^3\right) = \frac{1}{243\pi}(2\pi)\frac{3^6}{6} = 1 \quad .$$

7. (10 points) Compute the line integral

$$\int_C \frac{4\sqrt{3}\, x\, y\, z}{3}\, ds \quad ,$$

where C is the line-segment joining $(0,0,0)$ and $(1,1,1)$

The **type** of the answer is: Number

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ans. 1

The **parametric equation** of X is

$$\langle 0,0,0\rangle + t(\langle 1,1,1\rangle - \langle 0,0,0\rangle) = < t,t,t > \quad , 0 \leq t \leq 1 \quad .$$

In other words $x = t, y = t, z = t$ and $\mathbf{r}(t) = \langle t,t,t\rangle$.
We have

$$\mathbf{r}'(t) = \langle 1,1,1\rangle \quad .$$

so $||\mathbf{r}'(t)|| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$.
Hence our line integral is

$$\int_0^1 \frac{4\sqrt{3}\,(t)\,(t)\,(t)}{3}\sqrt{3}\,dt \; = \int_0^1 4t^3\,dt = \frac{4t^4}{4}\Big|_0^1 = t^4\Big|_0^1 = 1^4 - 0^4 = 1 - 0 = 1 \quad .$$

Comment: Some people confuse **scalar** line-integral (this problem) and **vector-field** line-integral. Be careful!

8. (10 points) Compute

$$\int_0^3 \int_{\sqrt{y/3}}^1 e^{x^3}\, dx\, dy \quad .$$

(Hint: Not even Dr. Z. can do $\int e^{x^3}\, dx$, so you must be clever, and first change the order of integration.)

The **type** of the answer is: Number

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ans. $e - 1$.

The Type II description of our 2D region is

$$\{(x, y) \,|\, 0 \le y \le 3 \quad , \quad \sqrt{y/3} \le x \le 1\,\} \quad .$$

This is a "triangular" region (except that one of its sides, from $(0,0)$ to $(1,3)$ is not a straight line but rather the curve $x = \sqrt{y/3}$, alias $y = 3x^2$.
By drawing a little diagram, it is seen that the Type I description of the same region os

$$\{(x, y) \,|\, 0 \le x \le 1 \quad , \quad 0 \le y \le 3x^2\,\} \quad .$$

Hence our double-integral equals the double-integral

$$\int_0^1 \int_0^{3x^2} e^{x^3}\, dy\, dx \quad .$$

The **inner integral** is

$$\int_0^{3x^2} e^{x^3}\, dy \;=\; e^{x^3} \int_0^{3x^2} \; dy \;=\; e^{x^3} \cdot \left(y \Big|_{y=0}^{y=3x^2} \right) \;=\; e^{x^3} \cdot (3x^2 - 0) \;=\; 3x^2 e^{x^3} \quad .$$

The **inner integral** is

$$\int_0^1 3x^2 e^{x^3}\, dx \quad .$$

Doing the substitution $u = x^3$, we get $3x^2\, dx = du$, and our integral becomes (since $0^3 = 0$ and $1^3 = 1$)

$$\int_0^1 e^u\, du \;=\; e^u \Big|_0^1 = e^1 - e^0 \;=\; e - 1 \quad .$$

9. (10 points) Compute the volume integral

$$\int\int\int_E \frac{5\,(x^2+y^2+z^2)}{4\pi}\,dV \quad ,$$

where

$$E = \{(x,y,z)\,|\,x^2\;+\;y^2\;+\;z^2 \le 1\} \quad .$$

The **type** of the answer(s) is: Number

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ans. 1

The region is a sphere, center the origin, radius 1. Hence the most natural coordinate system is that of **spherical cooridantes**. The region is

$$\{(\rho,\phi,\theta)\,|\,0\le\rho\le 1 \quad , \quad 0\le\phi\le\pi \quad , \quad 0\le\theta\le 2\pi\} \quad .$$

Since $x^2+y^2+z^2=\rho^2$ and $dV=\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$, our volume integral equals

$$\frac{5}{4\pi}\int_0^1\int_0^\pi\int_0^{2\pi}\rho^2\rho^2\sin\phi\,d\theta\,d\phi\,d\rho \quad = \frac{5}{4\pi}\int_0^1\int_0^\pi\int_0^{2\pi}\rho^4\sin\phi\,d\theta\,d\phi\,d\rho \quad .$$

Since the integrand is **separable** and the limits of integrations are **numbers**, we can simplify it and write it as product of three calc2 integrals

$$\frac{5}{4\pi}\left(\int_0^1 \rho^4\,d\,\rho\right)\cdot\left(\int_0^\pi \sin\phi\,d\,\phi\right)\cdot\left(\int_0^{2\pi} d\theta\right)\cdot$$

$$= \frac{5}{4\pi}\left(\frac{\rho^5}{5}\Big|_0^1\right)\cdot\left(-\cos\phi\Big|_0^\pi\right)\cdot(2\pi)$$

$$= \frac{5}{4\pi}\frac{1}{5}(-(-1)-(-1))(2\pi) \;=\; 1 \quad .$$

Comment: When you do the 'translation' from rectangular to spherical, remember that $x^2+y^2+z^2=\rho^2$. Some people did it the 'long way', using $x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$,, $z=\rho\cos\phi$, and wasted a long time.

10. (10 points) Find $\bigtriangledown \cdot \mathbf{F}$ if

$$\mathbf{F} \,=\, \langle\, \sin(xy)\,,\, \sin(yz)\,,\, \sin(xz)\,\rangle \quad .$$

The **type** of the answer is: multivariable function

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ans. $y\cos(xy) \,+\, z\cos(yz) \,+\, x\cos(xz) \quad .$

$$\bigtriangledown \cdot \mathbf{F} = \frac{\partial}{\partial x}(\sin(xy)) \,+\, \frac{\partial}{\partial y}(\sin(yz)) \,+\, \frac{\partial}{\partial z}(\sin(yz)) \quad .$$

$$= y\cos(xy) \,+\, z\cos(yz) \,+\, x\cos(xz) \quad .$$

Comment: $\bigtriangledown \cdot \mathbf{F}$ is another name for *div* $\mathbf{F}$. Do not confuse with $\bigtriangledown \times \mathbf{F}$ that is also denoted by *curl* $\mathbf{F}$. Also do not confuse with the **gradient**, $\bigtriangledown f$, that only makes sense for f a (single) function. $\bigtriangledown\mathbf{F}$ is **nonsense**.