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Solutions to MATH 251 (4,6,7), Dr. Z. , Exam 1, Thurs., Oct. 16, 2017, SEC 118

1. (10 pts.) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ **at the point** $(1,1,1)$ if $z(x,y)$ is given implictly by the equation

$$x^3 + y^3 + z^3 - 2xyz = 1 \quad .$$

The **type** of the answers is: **numbers**.

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ans. $\frac{\partial z}{\partial x}(1,1) \,=\, -1$ $\qquad\qquad$ $\frac{\partial z}{\partial y}(1,1) \,=\, -1$

Differentiating with respect to x we get, abbreviating $z' = \frac{\partial z}{\partial x}$

$$3x^2 + 3z^2 z' - 2y(xz' + z) \,=\, 0 \quad .$$

Plugging-in $x = 1, y = 1, z = 1$ we get

$$3 \cdot 1^2 + 3 \cdot 1^2 z'(1,1) - 2 \cdot (1 \cdot z'(1,1) + 1) \,=\, 0 \quad .$$

So

$$3 + 3z'(1,1) - 2z'(1,1) - 2 \,=\, 0 \quad ,$$

gives $z'(1,1) = -1$, hence $\frac{\partial z}{\partial x}(1,1) \,=\, -1$.

Next, Differentiating with respect to y we get, abbreviating $z' = \frac{\partial z}{\partial y}$

$$3y^2 + 3z^2 z' - 2x(yz' + z) \,=\, 0 \quad .$$

Plugging-in $x = 1, y = 1, z = 1$ we get

$$3 \cdot 1^2 + 3 \cdot 1^2 z'(1,1) - 2 \cdot (1 \cdot z'(1,1) + 1) \,=\, 0 \quad .$$

So

$$3 + 3z'(1,1) - 2z'(1,1) - 2 \,=\, 0 \quad ,$$

gives $z'(1,1) = -1$, hence $\frac{\partial z}{\partial y}(1,1) \,=\, -1$.

2. (10 points) Find $\frac{\partial h}{\partial r}$ at $(q,r)=(2,1)$ where $h(u,v)=ue^{v^2}$, $u=q^3+q$, $v=q^2r^3$

The **type** of the answer is: **number**

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ans. $960e^{16}$.

By the **chain rule**:

$$\frac{\partial h}{\partial r} = \frac{\partial h}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial h}{\partial v}\frac{\partial v}{\partial r} \quad .$$

$$e^{v^2}\cdot 0 + (2uv)e^{v^2}(3q^2r^2) = (2uv)e^{v^2}(3q^2r^2) \quad .$$

When $q=2$ and $r=1$, we have

$$u = 2^3+2 = 10 \quad , \quad v = 2^2\cdot 1^3 = 4 \quad .$$

Hence $\frac{\partial h}{\partial r}$ evaluated at $(q,r)=(2,1)$ equals

$$2\cdot 10\cdot 4\cdot e^{4^2}\cdot 3\cdot 2^2\cdot 1^2 = 960e^{16} \quad .$$

3. (10 points) Find the directional derivative of $f(x, y, z) = x^3y^4z^5$ at $P = (1, -1, 1)$ in the direction pointing from the point P to the point $Q = (1, 2, 2)$. (Hint: first find the vector $\mathbf{PQ}$.)

The **type** of the answer is: **number** .

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ans. $-\frac{7}{\sqrt{10}}$ or (a little nicer) $-\frac{7\sqrt{10}}{10}$.

$$\nabla f = \langle 3x^2y^4z^5, 4x^3y^3z^5, 5x^3y^4z^4 \rangle \quad .$$

Hence

$$\nabla f(1, -1, 1) = \langle 3, -4, 5 \rangle \quad .$$

$$\mathbf{PQ} = \langle 1, 2, 2 \rangle - \langle 1, -1, 1 \rangle = \langle 0, 3, 1 \rangle \quad .$$

The **unit vector** is

$$\mathbf{u} = \frac{\langle 0, 3, 1 \rangle}{||\langle 0, 3, 1 \rangle||} = \frac{\langle 0, 3, 1 \rangle}{\sqrt{0^2 + 3^2 + 1^2}} = \frac{1}{\sqrt{10}} \langle 0, 3, 1 \rangle \quad .$$

Hence the desired **diretional derivative** is

$$\frac{1}{\sqrt{10}} \langle 3, -4, 5 \rangle . \langle 0, 3, 1 \rangle = \frac{1}{\sqrt{10}} (3 \cdot 0 + (-4) \cdot 3 + 5 \cdot 1) = -\frac{7}{\sqrt{10}} \quad .$$

4. (10 points) Find an equation of the tangent plane to the following surface at the given point

$$xy + 2yz + 3xz \; = \; 6 \quad , (1, 1, 1) \quad .$$

The **type** of the answer is: **Equation of a plane**

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ans. $4\,x \; + \; 3\,y \; + \; 5\,z \; = \; 12 \quad .$

$$\nabla f = \langle y + 3z, x + 2z, 2y + 3x \rangle \quad ,$$

$$\nabla f(1, 1, 1) = \langle 4, 3, 5 \rangle \quad .$$

Eq. of tangent plane:

$$4(x - 1) + 3(y - 1) + 5(z - 1) \; = \; 0 \quad .$$

Simplifying:

$$4\,x \; + \; 3\,y \; + \; 5\,z \; = \; 12 \quad .$$

5. (10 points) Compute $f_{xy}(1,1)$ if $f(x,y) = x^5 \ln(x+y)$.

The **type** of the answer is: **number** .

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ans. $\frac{9}{4}$.

$$
\begin{aligned}
f_x &= x^5 \cdot \frac{1}{x+y} + 5x^4 \ln(x+y) \quad . \\
f_{xy} &= x^5 \cdot \frac{-1}{(x+y)^2} + 5x^4 \cdot \frac{1}{x+y} \quad . \\
f_{xy}(1,1) &= 1^5 \cdot \frac{-1}{(1+1)^2} + 5 \cdot 1^4 \cdot \frac{1}{1+1} = -\frac{1}{4} + \frac{5}{2} = \frac{9}{4} \quad .
\end{aligned}
$$

6. (10 points) Use the linearization of $f(x,y,z) = \sqrt{2x+3y+4xz}$ to approximate $f(1.01, 0.99, 1.02)$.

The **type** of the answer is: **number** .

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ans. $3\frac{11}{600}$ or $\frac{1811}{600}$.

Here the "nice" point (a,b,c) is $(1,1,1)$, so $a=1, b=1, c=1$. Since

$$f(x,y,z) = (2x+3y+4xz)^{1/2} \quad ,$$

we have

$$f_x = \frac{1}{2} \cdot (2x+3y+4xz)^{-1/2} \cdot (2+4z)$$

$$f_y = \frac{1}{2} \cdot (2x+3y+4xz)^{-1/2} \cdot (3)$$

$$f_z = \frac{1}{2} \cdot (2x+3y+4xz)^{-1/2} \cdot (4x)$$

Hence

$$f_x(1,1,1) = \frac{1}{2} \cdot (9)^{-1/2} \cdot (6) \ = \ 1$$

$$f_y = \frac{1}{2} \cdot (9)^{-1/2} \cdot (3) \ = \ \frac{1}{2}$$

$$f_z = \frac{1}{2} \cdot (9)^{-1/2} \cdot (4) = \frac{2}{3} \quad .$$

$$f(a,b,c) = f(1,1,1) = \sqrt{9} = 3 \quad .$$

Hence the **Linearization** of the function at $(1,1,1)$ is

$$L(x,y,z) = 3 + 1 \cdot (x-1) + \frac{1}{2} \cdot (y-1) + \frac{2}{3} \cdot (z-1) \quad .$$

Finally

$$L(1.01, 0.99, 1.02) = 3 + 1 \cdot (1.01-1) + \frac{1}{2} \cdot (0.99-1) + \frac{2}{3} \cdot (1.02-1)$$

$$= 3 + 0.01 + \frac{1}{2} \cdot (-0.01) + \frac{2}{3} \cdot (0.02) = 3 + 0.01 \cdot (1 - \frac{1}{2} + \frac{4}{3}) = 3 + \frac{11}{600} = 3\frac{11}{600} = \frac{1811}{600} \quad .$$

7. (10 points, altogether) Do the following limits exist? If they do, find them. Explain!

a. (3 points)

$$\lim_{(x,y,z)\to(1,1,1)} \frac{\ln(x^2+y^2+z^2)}{x+y+z} = \frac{\ln 3}{3} \quad .$$

(Just plug-it-in!), so the limit exists and equals $\frac{\ln 3}{3}$.

b. (3 points)

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x+y+z}{x^3+y^3+z^3} \quad .$$

Does Not exist. When you go along the x axis (i.e. $y=0$ and $z=0$) it blows up $\lim x \to 0\frac{1}{x^2}$. That's enough to disqualify it.

c. (4 points)

$$\lim_{(x,y,z)\to(0,0,0)} (x+y+z)\,\sin(\frac{1}{x+y+z}) \quad .$$

The limit is zero. The limit of $x+y+z$ as (x,y,z) goes to $(0,0,0)$ is obviously 0. $\sin(\frac{1}{x+y+z})$ is "crazy" but bounded (it is always between -1 and 1) so the limit goes to 0 by the **squeezing theorem**.

8. (10 points) Find the local maximum and minimum point(s), and saddle points (if they exist) of the functions

$$f(x, y) \;=\; 6x^2 \;-\; 2x^3 \;+\; 3y^2 \;+\; 6xy \quad .$$

The **type** of the answer(s) is: **points** (or N/A)

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ans.

local maximum point(s): None

local minimum point(s): $(0, 0)$

saddle point(s): $(1, -1)$.

$$f_x = 12x - 6x^2 + 6y \quad , \quad f_y = 6y + 6x \quad .$$

For future reference

$$f_{xx} = 12 - 12x \quad , \quad f_{xy} = 6 \quad , \quad f_{yy} = 6 \quad .$$

We have to solve $f_x = 0, f_y = 0$, in other words

$$12x - 6x^2 + 6y = 0 \quad , \quad 6y + 6x = 0 \quad .$$

From the second equation, $y = -x$, and putting it in the first equation:

$$12x - 6x^2 - 6x = 0 \quad ,$$

so

$$6x - 6x^2 = 0 \quad ,$$

so

$$6x(1 - x) = 0 \quad .$$

We get **two** solutions $x = 0$ and $x = 1$. When $x = 0$, we have $y = -0 = 0$, and when $x = 1$, we have $y = -1$. So the two **candidates** are the points $(0, 0)$ and $(1, -1)$.
When $(x, y) = (0, 0)$,

$$f_{xx} = 12 \quad , \quad f_{xy} = 6 \quad , \quad f_{yy} = 6 \quad ,$$

and $D = 12 \cdot 6 - 6^2 = 36 > 0$, so it is either a local min or local max. Since $f_{xx} > 0$, it is a **local min**.
When $(x, y) = (1, -1)$,

$$f_{xx} = 0 \quad , \quad f_{xy} = 6 \quad , \quad f_{yy} = 6 \quad ,$$

and $D = 0 \cdot 6 - 6^2 = -36 < 0$, so it is a **saddle point**.

9. (10 points) A certain particle has acceleration

$$\mathbf{a}(t) = \langle\, e^t, -\sin t, -4\cos 2t \,\rangle \quad ,$$

and at $t = 0$ its velocity is $\langle 1, 1, 0\rangle$ and its position vector is $\langle 1, 0, 1\rangle$, find its velocity and position vector at time $t = \frac{\pi}{2}$.

The **type** of the answer(s) is:$3D$ vectors of numbers.

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ans.

velocity vector at $t = \frac{\pi}{2}$: $\langle e^{\pi/2}, 0, 0\rangle$.

position vector at $t = \frac{\pi}{2}$: $\langle e^{\pi/2}, 1, -1\rangle$.

$$\mathbf{v}\,(t) \;=\; \int \langle\, e^t, -\sin t, -4\cos 2t \,\rangle\, dt = \langle\, e^t, \cos t, -2\sin 2t \,\rangle + \mathbf{C} \quad .$$

But $\mathbf{v}(0) = \langle 1, 1, 0\rangle$ so $\mathbf{C} = \langle 0, 0, 0\rangle$, and

$$\mathbf{v}\,(t) \;=\; \langle\, e^t, \cos t, -2\sin 2t \,\rangle \quad .$$

next

$$\mathbf{r}\,(t) \;=\; \int \langle\, e^t, \cos t, -2\sin 2t \,\rangle\, dt = \langle\, e^t, \sin t, \cos 2t\rangle \,+ \mathbf{C}$$

(for another $\mathbf{C}$). Since $\mathbf{r}\,(0) \;=\; \langle 1, 0, 1\rangle$, once again $\mathbf{C} = \langle 0, 0, 0\rangle$. So

$$\mathbf{r}\,(t) \;=\; \langle\, e^t, \sin t, \cos 2t\rangle \quad .$$

Plugging-in $t = \frac{\pi}{2}$, gives

$$\mathbf{v}(\pi/2) = \langle\, e^{\pi/2}, \cos(\pi/2), -2\sin \pi \,\rangle = \langle e^{\pi/2}, 0, 0\rangle \quad ,$$

$$\mathbf{r}(\pi/2) = \langle\, e^{\pi/2}, \sin(\pi/2), \cos \pi \,\rangle = \langle e^{\pi/2}, 1, -1\rangle \quad ,$$

10. (10 points) Find an equation to the plane that passes through the points $(6,0,0)$, $(0,4,0)$, $(0,0,3)$.

The **type** of the answer is: Equation of plane

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ans. $2x + 3y + 4z = 12$

The easiest way to do it is without cross product.
An equation of a plane (that does not pass through the origin) may be written as

$$ax + by + cz = 1$$

where a, b, c are some numbers.
Since the point $(6,0,0)$ lies on our place, we have

$$6a + b \cdot 0 + c \cdot 0 = 1 \quad ,$$

so $a = \frac{1}{6}$.
Since the point $(0,4,0)$ lies on our place, we have

$$a \cdot 0 + b \cdot 4 + c \cdot 0 = 1 \quad ,$$

so $b = \frac{1}{4}$.
Since the point $(0,0,3)$ lies on our place, we have

$$a \cdot 0 + b \cdot 0 + c \cdot 3 = 1 \quad ,$$

so $c = \frac{1}{3}$. Hence an equation of the plane is

$$\frac{x}{6} + \frac{y}{4} + \frac{z}{3} = 1 \quad .$$

Multiplying by 12, we get

$$2x + 3y + 4z = 12 \quad .$$