Solutions to MATH 251 (04,06,07), Dr. Z. , Final Exam ,Tue., Dec. 19, 2017, SEC 118, 12:00-3:00pm

1. (12 points) Compute the line-integral

$$\int_C 7y\,dx + 3x\,dy \quad ,$$

where C is the circle $x^2 + y^2 = 100$ traveled in the clockwise direction.

Ans.: 400π

First Way (directly): The parametric equation of the circle $x^2 + y^2 = 100$ is

$$x = 10\cos t \quad , \quad y = 10\sin t \quad , 0 \leq t \leq 2\pi \quad .$$

Hence

$$dx = -10\sin t\,dt \quad , \quad dy = 10\cos t\,dt \quad .$$

Hence (with the default **positive(counter-clockwise)** direction)

$$\int_C 7y\,dx + 3x\,dy \;=\; \int_0^{2\pi} 7(10\sin t)(-10\sin t)\,dt + 3(10\cos t)(10\cos t)\,dt =$$

$$= -700\int_0^{2\pi} \sin^2 t\,dt \;+\; 300\int_0^{2\pi} \cos^2 t\,dt \quad .$$

Now

$$\int_0^{2\pi} \sin^2 t\,dt = \int_0^{2\pi} \frac{1-\cos 2t}{2}\,dt = \left(\frac{t}{2} - \frac{\sin 2t}{4}\right)\Big|_0^{2\pi} = \frac{2\pi - 0}{2} = \pi \quad .$$

Also

$$\int_0^{2\pi} \cos^2 t\,dt = \int_0^{2\pi} \frac{1+\cos 2t}{2}\,dt = \left(\frac{t}{2} - \frac{\sin 2t}{4}\right)\Big|_0^{2\pi} = \frac{2\pi - 0}{2} = \pi \quad .$$

Going back above we get $-700\pi + 400\pi = -400\pi$.
But the direction specified was **clockwise** hence, we have to multiply by -1, getting 400π.

Second Way: Use **Green's Theorem**

$$\int_C P\,dx + Q\,dy \;=\; \int\int \left(\frac{\partial Q}{\partial x} \;-\; \frac{\partial P}{\partial y}\right)\,dA \quad ,$$

where D is the interior region (the disc). Here $P = 7y$ and $Q = 3x$, so

$$\int_C 7y\,dx + 3x\,dy \;=\; \int\int_D \frac{\partial}{\partial x}(3x) \;-\; \frac{\partial}{\partial y}(7y)$$

$$\int\int_D (3-7)\,dA \;=\; \int\int_D (-4)\,dA \;=\; -4\left(\int\int_D dA\right) = -4 \cdot Area(D) = -4(\pi \cdot 10^2) = -400\pi \quad .$$

Finally, multiply by -1, to get 400π.

Comments: Full credit to either method! People who forgot to multiply by -1 only got half of the points.

2. (12 points) Find an equation of the tangent plane to the surface

$$z \;=\; x^2 \;+ 3xy \;+\; y^2 \quad ,$$

at the point $(1, 1, 5)$.

Ans.: $z = 5x + 5y - 5 \quad .$

$$\frac{\partial z}{\partial x} = 2x + 3y \quad , \quad \frac{\partial z}{\partial y} = 3x + 2y \quad .$$

At $(x, y) = (1, 1)$ we have

$$\frac{\partial z}{\partial x}(1, 1) = 5 \quad , \quad \frac{\partial z}{\partial y}(1, 1) = 5 \quad .$$

Hence the equation of the tangent plane at the given point is

$$z - 5 = 5(x - 1) + 5(y - 1) \quad .$$

Expanding and adding 5 to both sides gives the answer.

3. (12 points) Find the absolute maximum value and the absolute minimum value of the function $f(x, y) = x^2\, y$ in the region

$$\{(x, y)\,|\, 0 \leq x \leq 1\,,\, 0 \leq y \leq 1 - x\,\}.$$

Absolute minimum value: 0 .

Absolute maximum value: $\frac{4}{27}$.

$$f_x = 2xy \quad , f_y = x^2 \quad .$$

Local max and min are when $2xy = 0$ and $x^2 = 0$ so $x = 0$ and y anything. At any rate, the value of the function is 0 at all these points.
The triangle has three sides. The value of the function along the side from $(0, 0)$ to $(1, 0)$ is 0 so both max and min values on this side are 0. Similarly for the side from $(0, 0)$ to $(0, 1)$. The only interesting side is the side from $(1, 0)$ to $(0, 1)$, i.e., along the line-segment $y = 1 - x$, $0 \leq x \leq 1$. Plugging $y = 1 - x$ we have to do the calc1 problem:
Find the abs. max and abs. min values of the function $g(x) = x^2(1 - x)$ on the closed interval $0 \leq x \leq 1$. Since $g(x) = x^2 - x^3$, we get $g'(x) = 2x - 3x^2$. Setting this to 0, gives, $x = 0$ and $x = \frac{2}{3}$. The value when $x = \frac{2}{3}$ is $(\frac{2}{3})^2 \cdot (1 - \frac{2}{3}) = \frac{4}{27}$. In addition, we have to check the end-points $x = 0$ and $x = 1$ giving 0.
In the **final contest**, the candiates are 0 and $\frac{4}{27}$. So the abs. min. value is 0 and abs. max. value is $\frac{4}{27}$.

4. (12 points) Compute $f_{xxyz}(0,0,0)$ (in other words $\frac{\partial^4}{\partial x^2 \partial y \partial z} f(x,y,z)|_{x=0,y=0,z=0}$) if

$$f(x,y,z) = \sin(\, x^2 \; + \; y \; + \; z\,) \quad .$$

Ans.: -2 .

$$f_x = 2x\cos(x^2+y+z) \quad ,$$

$$f_{xx} = 2\cos(x^2+y+z) + 2x(-2x)\sin(2x^2+y+z) \;=\; 2\cos(x^2+y+z) - 4x^2\,\sin(2x^2+y+z) \quad ,$$

$$f_{xxy} = -2\sin(x^2+y+z) - 4x^2\cos(x^2+y+z)$$

$$f_{xxyz} = -2\cos(x^2+y+z) + 4x^2\sin(x^2+y+z) \quad .$$

Plugging-in $x = 0, y = 0, z = 0$ we get

$$f_{xxyz}(0,0,) = -2\cos(0^2+0+0) + 4\cdot(0^2)\sin(0^2+0+0) = -2 \quad .$$

5. (12 points) Find $\frac{\partial z}{\partial y}$ at the point $(1,1,1)$ if (x,y,z) are related by:

$$xy + xz + yz + x^2y^2z^2 = 4 \quad .$$

Ans.: -1 .

First Way: use **implicit differentiation**. Differentiate the relation w.r.t. to y, using the addition, product, and chain rules.

$$x + x \cdot z_y + z + y \cdot z_y + x^2y^2(2z) \cdot (z_y) + x^2(2y)z^2 = 0 \quad .$$

Now **plug-in**, $x = 1, y = 1, z = 1$, getting

$$1 + 1 \cdot z_y(1,1) + 1 + 1 \cdot z_y(1,1) + 1^2 \cdot 1^2 \cdot (2 \cdot 1) \cdot z_y(1,1) + 1^2 \cdot (2 \cdot 1) \cdot (1^2) = 0 \quad .$$

Hence

$$4 + 4z_y(1,1) = 0 \quad ,$$

giving $z_y(1,1) = -1$.

Second Way: Use the formula $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$, with $F = xy + xz + yz + x^2y^2z^2 - 4$, getting

$$\frac{\partial z}{\partial y} = -\frac{x + z + 2x^2yz}{x + y + 2x^2y^2z} \quad .$$

Not plug-in $x = 1, y = 1, z = 1$, to get

$$z_y(1,1) = - - \frac{1 + 1 + 2 \cdot 1^2 \cdot 1 \cdot 1}{1 + 1 + 2 \cdot 1^2 \cdot 1^2 \cdot z} = -\frac{4}{4} = -1 \quad . \quad .$$

6. (12 points) Find an equation for the plane that contains both the line

$$x = 1 + t\,,\ y = 2 + t\,,\ z = 3 + t \quad (-\infty < t < \infty) \quad ,$$

and the line

$$x = -t\,,\ y = 1 + t\,,\ z = 2 + t \quad (-\infty < t < \infty) \quad .$$

Ans.: $y - z = -1$, OR $z = y + 1$.

The first line has direction vector $\langle 1, 1, 1\rangle$ and the second line has direction vector $\langle -1, 1, 1\rangle$. A **normal vector** to the plane is

$$\langle 1, 1, 1\rangle \times \langle -1, 1, 1\rangle = \langle 0, -2, 2\rangle \quad .$$

(You do it!) We also need a point. Plugging-in $t = 0$ into the second line, we get the point $(0, 1, 2)$. Hence an equation of the plane is

$$0 \cdot (x - 0) + (-2) \cdot (y - 1) + 2 \cdot (z - 2) = 0 \quad .$$

Dividing by -2:

$$(y - 1) - (z - 2) = 0 \quad .$$

and simplifying, we get $y - z = -1$ (or $y - z + 1 = 0$).

7. (12 points) A certain particle has acceleration given by

$$\mathbf{a}(t) = \langle\, -4\,\sin 2t\,,\, -4\,\cos 2t\,,\, 9e^{3t}\,\rangle \quad .$$

If its velocity at $t = 0$ is $\langle 2, 0, 3\rangle$ and its position at $t = 0$ is $\langle 0, 1, 1\rangle$, finds its position at the time $t = \frac{\pi}{4}$.

Ans.: $(1\,,\,0\,,\,e^{3\pi/4}) \quad .$

$$\mathbf{v}(t) = \int \langle\, -4\,\sin 2t\,,\, -4\,\cos 2t\,,\, 9e^{3t}\,\rangle\, dt = \langle\, 2\,\cos 2t\,,\, -2\,\sin 2t\,,\, 3e^{3t}\,\rangle \,+\, \mathbf{C} \quad .$$

Substituting $t = 0$ gives $\mathbf{C} = \mathbf{0}$ hence

$$\mathbf{v}(t) = \langle\, 2\,\cos 2t\,,\, -2\,\sin 2t\,,\, 3e^{3t}\,\rangle \quad .$$

$$\mathbf{r}(t) = \int \langle\, 2\,\cos 2t\,,\, -2\,\sin 2t\,,\, 3e^{3t}\,\rangle\, dt \;=\; \langle\, \sin 2t\,,\, \cos 2t\,,\, e^{3t}\,\rangle \,+\, \mathbf{C} \quad .$$

Substituting $t = 0$ gives $\mathbf{C} = \mathbf{0}$. Hence

$$\mathbf{r}(t) = \langle\, \sin 2t\,,\, \cos 2t\,,\, e^{3t}\,\rangle \quad .$$

Substituting $t = \frac{\pi}{4}$ gives

$$\mathbf{r}(\frac{\pi}{4}) = \langle\, \sin(2 \cdot \frac{\pi}{4})\,,\, \cos(2\frac{\pi}{4})\,,\, e^{3 \cdot \frac{\pi}{4}}\,\rangle \quad .$$

$$= \langle\, \sin(\frac{\pi}{2})\,,\, \cos(\frac{\pi}{2})\,,\, e^{\frac{3\pi}{4}}\,\rangle \;=\; \langle\, 1\,,\, 0\,,\, e^{\frac{3\pi}{4}}\,\rangle \quad .$$

That is the position **vector**, converting it to a **point** we change $\langle \ldots \rangle$ to $(\ldots)$.

8. (12 points) Compute the (scalar-function) line-integral

$$\int_C (x \, + \, y \, + \, 2\,z)\, ds$$

where the curve C is given by the parametric equation:

$$\mathbf{r}(t) = \langle\, t, 2t, 2t \rangle \quad , \quad 0 \le t \le 1 \quad .$$

Ans.: $\frac{21}{2}$.

$$\mathbf{r}'(t) = \langle\, 1, 2, 2 \rangle \quad ,$$

Hence

$$||\mathbf{r}'(t)|| = ||\langle\, 1, 2, 2 \rangle|| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3 \quad ,$$

hence $ds = 3\,dt$. We have

$$\int_C (x + y + 2\,z)\, ds \; = \; \int_0^1 (t{+}2t{+}2(2t))\cdot(3)\, dt = \int_0^1 7t\cdot(3)\, dt = 21 \int_0^1 t\, dt = 21 \frac{t^2}{2}\Big|_0^1 = 21 \frac{1^2 - 0^2}{2} = \frac{21}{2} \quad .$$

9. (12 points)

If

$$\lim_{(x,y,z)\to(1,1,1)} f(x,y,z) \;=\; 1 \quad , \qquad \lim_{(x,y,z)\to(1,1,1)} g(x,y,z) \;=\; 2$$

compute

$$\lim_{(x,y,z)\to(1,1,1)} \sin(\frac{\pi}{3} f(x,y,z)) \cos(\frac{\pi}{4} g(x,y,z)$$

Ans.: 0 .

Since sin and cos are **continuous** the answer is simply

$$\sin(\frac{\pi}{3}\cdot 1)\cos(\frac{\pi}{4}\cdot 2) = \sin(\pi/3)\cdot\cos(\pi/2) = \frac{\sqrt{3}}{2}\cdot 0 = 0 \quad .$$

10. (12 points) Compute

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} \quad ,$$

where

$$\mathbf{F} = \langle x^2 + \sin(y+z)\, ,\, y^2 + xz^3\, ,\, z^2 + e^{xy} \rangle \quad ,$$

and where S is the boundary (consisting of all six faces) of the cube

$$\{(x,y,z)\,|\,0 \leq x,y,z \leq 1\}$$

with the normal pointing **outward**.

Ans.: 3 .

We use the **Divergence Theorem**.

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} \;=\; \int\int\int_E div\mathbf{F}\, dV \quad ,$$

where E is the box $\{(x,y,z)\,|\,0 \leq x,y,z \leq 1\}$.
We have

$$div\mathbf{F} = \frac{\partial}{\partial x}(x^2 + \sin(y+z)) + \frac{\partial}{\partial y}(y^2 + xz^3) + \frac{\partial}{\partial z}(z^2 + e^{xy}) = 2x + 2y + 2z \quad .$$

We need

$$\int_0^1 \int_0^1 \int_0^1 (2x + 2y + 2z)\, dx\, dy\, dz \quad .$$

The **inner integral** is

$$\int_0^1 (2x + 2y + 2z)\, dx \;=\; \left(x^2 + (2y + 2z)x\right)\Big|_0^1 = 1 + 2y + 2z \quad .$$

The **middle integral** is

$$\int_0^1 (1 + 2y + 2z)\, dy = \left(y + y^2 + 2zy\right)\Big|_0^1 = (1-0) + (1^2 - 0^2) + 2z(1-0) = 2 + 2z \quad .$$

The **outer integral** is

$$\int_0^1 (2 + 2z)\, dz = \left(2z + z^2\right)\Big|_0^1 = 2(1-0) + (1^2 - 0^2) = 3 \quad .$$

11. (12 points) By finding a function f such that $\mathbf{F} = \bigtriangledown f$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

$$\mathbf{F}(x, y, z) \;=\; \langle\, 2e^{2x+3y+4z} \,,\, 3e^{2x+3y+4z} \,,\, 4e^{2x+3y+4z} \,\rangle \quad ,$$

$$C : x = t \quad , \quad y = 2t \quad , \quad z = t^2 \quad , \quad 0 \leq t \leq 1 \quad .$$

Ans: $\quad e^{12} - 1 \quad .$

$$f(x, y, z) = \int 2e^{2x+3y+4z}\, dx = e^{2x+3y+4z} + g(y, z) \quad ,$$

where $g(y, z)$ is **to be determined**. Diff. w.r.t to y, we get

$$f_y = 3\, e^{2x+3y+4z} + g_y(y, z) \quad .$$

Hence

$$3\, e^{2x+3y+4z} = 3\, e^{2x+3y+4z} + g_y(y, z) \quad ,$$

hence $g_y(y, z) = 0$, hence $g(y, z) = h(z)$ where $h(z)$ is to be determined, i.e.

$$f(x, y, z) = e^{2x+3y+4z} + h(z) \quad .$$

Differentiating both sides w.r.t. to z, we get

$$f_z = 4e^{2x+3y+4z} + h'(z) \quad .$$

Hence

$$4e^{2x+3y+4z} = 4e^{2x+3y+4z} + h'(z) \quad ,$$

hence $h'(z) = 0$ and so $h(z)$ is a constant, that we can make 0. Hence

$$f(x, y, z) = e^{2x+3y+4z} \quad .$$

The **beginning** of the path is the point $(0, 0, 0)$ and the **end** is the point $(1, 2, 1)$. By the **Fundamental Theorem of Line Integrals** the answer is

$$f(1, 2, 1) - f(0, 0, 0) = e^{2\cdot 1+3\cdot 2+4\cdot 1} - e^0 = 2^{12} - 1 \quad .$$

Comment: It would have been OK to find $f(x, y, z)$ by **inspection**, but then you should have checked that indeed $f_x = 2e^{2x+3y+4z}$, $f_y = 3e^{2x+3y+4z}$, $f_z = 4e^{2x+3y+4z}$.

12. (12 points) Evaluate the line integral

$$\int_C 5y\,dx + 5x\,dy + 6z\,dz \quad ,$$

where $C : x = t^2\,,\ y = t\,,\ z = t^2\,,\ 0 \le t \le 1$.

Ans.: 8 .

First Way (directly):

$$dx = 2t\,dt\,,\ dy = dt\,,\ dz = 2t\,dt \quad .$$

Hence

$$\int_C 5y\,dx + 5x\,dy + 6z\,dz = \int_0^1 5t\,(2t\,dt) + 5t^2\,(dt) + 6(t^2)\,(2t)\,dt = \int_0^1 (10\,t^2 + 5\,t^2 + 12\,t^3)\,dt$$

$$\int_0^1 (15\,t^2\ +\ 12\,t^3)\,dt = \left(5t^3 + 3t^4\right)\Big|_0^1 = 5\cdot(1^3 - 0^3) + 3\cdot(1^4 - 0^4) = 5 + 3 = 8 \quad .$$

Second Way (Fundamental Theorem of Line Integrals). By inspection (or the long way) this a conservative vector field, with potential function $f(x,y,z) = 5xy + 3z^2$. The starting point is $(0,0,0)$ and the end-point is $(1,1,1)$, hence the answer is $f(1,1,1) - f(0,0,0) = 5\cdot 1\cdot 1 + 3\cdot 1^2 \cdot 8\ -\ 0\ = 8$.

13. (12 points) Evaluate

$$\int\int\int_E \frac{1}{\sqrt{x^2+y^2+z^2}}\, dV \quad ,$$

where E is the hemisphere

$$\{(x,y,z)\,|\,x^2+y^2+z^2 \le 100, z<0\} \quad .$$

Ans.: 100π .

Converting to **spherical coordinates** (since $z<0$ we have $\pi/2 \le \phi \le \pi$.

$$\int_0^{10}\int_{\pi/2}^{\pi}\int_0^{2\pi} \frac{1}{\rho}\rho^2 \sin\phi d\theta d\phi d\rho \;=\; \int_0^{10}\int_{\pi/2}^{\pi}\int_0^{2\pi} \rho \sin\phi d\theta d\phi d\rho$$

$$= \left(\int_0^{10} \rho d\rho\right)\left(\int_{\pi/2}^{\pi} \sin\phi\, d\phi\right)\left(\int_0^{2\pi} d\theta\right)$$

$$= \left(\frac{\rho^2}{2}\Big|_0^{10}\right)\left(-\cos\phi\Big|_{\pi/2}^{\pi}\right)\cdot(2\pi) \;=\; \frac{10^2-0^2}{2}\cdot(--1-0)\cdot(2\pi) = 100\pi \quad .$$

14. (12 points) Evaluate the quadruple integral

$$\int\int\int\int_E 360\, x\, dV \quad ,$$

where

$$E = \{(x, y, z, w) \,|\, 0 \le w \le 1\,,\, 0 \le z \le w\,,\, 0 \le y \le z\,,\, 0 \le x \le y\} \quad .$$

Ans.: 3 .

We need

$$\int_0^1 \int_0^w \int_0^z \int_0^y 360\, x\, dx\, dy\, dz\, dw \quad .$$

The **innermost** integral is

$$\int_0^y 360x\, dx = (180x^2)\Big|_0^y = 180y^2 \quad .$$

The **first middle integral** is

$$\int_0^z 180y^2\, dy = 60y^3\Big|_0^z = 60z^3 \quad .$$

The **second middle integral** is

$$\int_0^w 60z^3\, dz = 15z^4\Big|_0^w = 15w^4 \quad .$$

The **outer integral** is

$$\int_0^1 15\, w^4\, dw = 3\, w^5\Big|_0^1 = 3(1^5 - 0^5) = 3 \quad .$$

15. (12 points) Find the Jacobian of the transformation from (u, v)-space to (x, y)-space.

$$x = 3\sin(2u + v) \quad , \quad y = u + v + \cos(u + v) \quad ,$$

at the point $(u, v) = (0, 0)$.

Ans.: 3 .

$$x_u = 6\cos(2u + v) \quad , \quad x_v = 3\cos(2u + v) \quad ,$$

$$y_u = 1 - \sin(u + v) \quad , \quad y_v = 1 - \sin(u + v) \quad .$$

$$J = \det \begin{pmatrix} 6\cos(2u + v) & 3\cos(2u + v) \\ 1 - \sin(u + v) & 1 - \sin(u + v) \end{pmatrix} \quad .$$

Plugging-in $u = 0$ and $v = 0$ we get

$$J(0, 0) = \det \begin{pmatrix} 6 & 3 \\ 1 & 1 \end{pmatrix} = (6)(1) - (3)(1) = 3 \quad .$$

16. (12 points) Find the local maximum and minimum **points** and saddle point(s) of the function $f(x, y) = x^3 + y^2 - 6xy$

Local maximum points(s): None .

Local minimum points(s): $(6, 18)$.

saddle point(s): $(0, 0)$.

$$f_x = 3x^2 - 6y \quad , \quad f_y = 2y - 6x \quad .$$

Solving $3x^2 = 6y$, and $2y = 6x$ gives $y = 3x$ and $y = x^2/2$ hence $3x - x^2/2 = 0$ hence $6x - x^2 = 0$ hence $x(6 - x) = 0$ and we get **two** candidates (since when $x = 0$, $y = 0$, and when $x = 6$, $y = 18$.

$$(0, 0) \quad , \quad (6, 18) \quad .$$

Also

$$f_{xx} = 6x \quad , \quad f_{xy} = -6 \quad , \quad f_{yy} = 2 \quad ,$$

Hence the **discriminant** is $D = (6x)(2) - (-6)^2 = 12x - 36$.
at $(0, 0)$, $D = -36$ is **negative**, hence it is a **saddle** point.
at $(6, 18)$, $D = 12 \cdot 6 - 36 = 36$ is **positive**, hence, since $f_{xx} = 36$ is also positive is is a **local minimum point**.

17. (8 points) Use the Divergence Theorem to calculate the surface integral $\int\int_S \mathbf{F} \cdot d\,\mathbf{S}$, where

$$\mathbf{F}(x,y,z) = \langle\, x+y \; y+z \,,\, x+z \,\rangle \quad ,$$

where S is the sphere (center $(1,-2,4)$ and radius 10), in other words the region in 3D space:

$$\{(x,y,z)\,|\,(x-1)^2 \,+\, (y+2)^2 \,+\, (z-4)^2 = 100\} \quad .$$

Ans.: 4000π .

Using the **Divergence theorem** this is (since $div\mathbf{F} = 1+1+1 = 3$)

$$\int\int\int_E 3\,dV \quad ,= 3\int\int\int_E dV \;=\; 3 \cdot Vol(E) \quad ,$$

where E is a ball of radius 10. But the volume of a ball of radius 10 is

$$\frac{4\pi}{3}10^3 = \frac{4000\pi}{3} \quad ,$$

giving the answer.