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MATH 251 (04,06,07), Dr. Z. , Final Exam ,Tue., Dec. 19, 2017, SEC 118, 12:00-3:00pm

WRITE YOUR FINAL ANSWER TO EACH PROBLEM IN THE INDICATED PLACE (right under the question)

Do not write below this line

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1. (out of 12)
2. (out of 12)
3. (out of 12)
4. (out of 12)
5. (out of 12)
6. (out of 12)
7. (out of 12)
8. (out of 12)
9. (out of 12)
10. (out of 12)
11. (out of 12)
12. (out of 12)
13. (out of 12)
14. (out of 12)
15. (out of 12)
16. (out of 12)
17. (out of 8)

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tot. (out of 200)

Important note: Unlike Exams 1 and 2, you are not required to state the type of the answer, and there is no credit for stating the type. But if the given answer is the **wrong type**, you would get 0 points.

Example: Find $f'(2)$ if $f(x) = x^3$. If you give the answer $3x^2$ instead of 12, you would get **zero** points!

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x,y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)\, dA\ .$$

1. (12 points) Compute the line-integral

$$\int_C 7y\,dx + 3x\,dy \quad ,$$

from answer neg.

where C is the circle $x^2 + y^2 = 100$ traveled in the clockwise direction.

Ans.: 400π

$$\int_C \overset{P}{7y}dx + \overset{Q}{3x}dy$$

$x = 10\cos t$

$y = 10\sin t$

$r = 10$

$0 \le t \le 2\pi$

$ds = \sqrt{(10\cos t)^2 + (10\sin t)^2}$

$= \sqrt{100} = 10$

$$\int_0^{2\pi} 7y(x'(t)) + 3x(y'(t))\,dt$$

$$= \int_0^{2\pi} 7(10\sin t)(-10\sin t) + 30\cos t(10\cos t)\,dt$$

$$= \int_0^{2\pi} (-700\sin^2 t + 300\cos^2 t)\,dt$$

$$= 250\sin(2x) - 200x \Big|_0^{2\pi}$$

$$= -400\pi(-1) = \boxed{400\pi}$$

2. (12 points) Find an equation of the tangent plane to the surface

$$z = x^2 + 3xy + y^2 \quad ,$$

at the point $(1, 1, 5)$.

Ans.: $5x + 5y - z = 5$

$z = x2 + 3xy + y2$ @ $(1, 1, 5)$ point of tangency (x_0, y_0, z_0)

$f_x = 2x + 3y$ @ $P = 5$

$f_y = 3x + 2y$ @ $P = 5$

$f_z = -1$ @ $P = -1$

$f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0) = 0$

$= \quad 5(x-1) + 5(y-1) + -1(z-5) = 0$

$5x - 5 + 5y - 5 - z + 5 = 0$

$\boxed{5x + 5y - z = 5}$

3. (12 points) Find the absolute maximum value and the absolute minimum value of the function $f(x, y) = x^2\, y$ in the region

$$\{(x, y)\,|\, 0 \le x \le 1\,,\, 0 \le y \le 1 - x\,\}.$$

Absolute minimum value: 0

Absolute maximum value: 1

$f(x,y) = x^2y \qquad \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1-x\}$

$f_x = 2xy = 0 \rightarrow x = 0,\ y = 0$

$f_y = x^2 = 0 \rightarrow x = 0$

$f(0,0) = 0$

on the leftside $x = 0,\ 0 \le y \le 1-x$

$f(0,y) = 0$

on the rightside $x = 1,\ 0 \le y \le 1-x$

$f(1,y) = y$

$f'(y) = 1$

on the down side $y = 0$

$f(x,0) = 0$

on the upside $y = 1-x$

$f(x, 1-x) = (1-x)(x^2)$

$= x^2 - x^3$

$f' = 2x - 3x^2 = 0 \qquad (1/3)\ \frac{4}{9} = \frac{4}{27} =$

$x(2-3x) = 0$

$x = 0, 2/3$

$f(0) = 0$ min

$f(\ldots) = 4/27$ max

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4. (12 points) Compute $f_{xxyz}(0,0,0)$ (in other words $\frac{\partial^4}{\partial x^2 \partial y \partial z} f(x,y,z)|_{x=0,y=0,z=0}$) if

$$f(x,y,z) = \sin(\, x^2 \,+\, y \,+\, z\,) \quad .$$

Ans.: 0

$f(x,y,z) = \sin(x^2+y+z)$

$f_x = 2x\cos(x^2+y+z)$

$f_{xx} = -4x^2\sin(x^2+y+z)$

$f_{yxx} = -4x^2\cos(x^2+y+z)$

$f_{yxxz} = 4x^2\sin(x^2+y+z)$

$= 4(0)^2\sin(0+0+0)$

$= \boxed{0}$

implicit differentiation

5. (12 points) Find $\frac{\partial z}{\partial y}$ at the point $(1, 1, 1)$ if (x, y, z) are related by:

$$xy + xz + yz + x^2y^2z^2 = 4 \quad .$$

Ans.: $\frac{dz}{dy}(1,1,1) = -1$

$$xy + xz + yz + x^2y^2z^2 = 4$$

$$x + z'x + z + z'y + 2yxz^2 + 2zz'y^2x^2 = 0$$

$$z'x + z'y + 2zz'y^2x^2 = -x - z - 2yxz^2$$

$$z'(x + y + 2zy^2x^2) = -x - z - 2yxz^2$$

$$z' = \frac{-x - z - 2yxz^2}{(x + y + 2zy^2x^2)}$$

$$z'(0,0,0) = \frac{-1 - 1 - 2}{(1 + 1 + 2)} = \frac{-4}{4} = -1$$

6. (12 points) Find an equation for the plane that contains both the line

↑

$$x = 1 + t\,,\; y = 2 + t\,,\; z = 3 + t \quad (-\infty < t < \infty) \quad ,$$

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and the line

$$x = -t\,,\; y = 1 + t\,,\; z = 2 + t \quad (-\infty < t < \infty) \quad .$$

Ans.: $P = \langle -t^2 + t + 1,\ t^2 + 2t + 2,\ t^2 + 3t + 3 \rangle$

$$A \begin{cases} x = 1 + t \\ y = 2 + t \\ z = 3 + t \end{cases} \longrightarrow \langle 1+t,\ 2+t,\ 3+t \rangle$$

$$B \begin{cases} x = -t \\ y = 1 + t \\ z = 2 + t \end{cases} \longrightarrow \langle -t,\ 1+t,\ 2+t \rangle$$

$P = \langle 1+t,\ 2+t,\ 3+t \rangle + \langle -t^2,\ t + t^2,\ 2t + t^2 \rangle$

$P = \langle -t^2 + t + 1,\ t^2 + 2t + 2,\ t^2 + 3t + 3 \rangle$

7. (12 points) A certain particle has acceleration given by

$$\mathbf{a}(t) = \langle -4 \sin 2t \, , \, -4 \cos 2t \, , \, 9e^{3t} \rangle \quad .$$

If its velocity at $t = 0$ is $\langle 2, 0, 3 \rangle$ and its position at $t = 0$ is $\langle 0, 1, 1 \rangle$, finds its position at the time $t = \cancel{2}\frac{\pi}{4}. = \frac{\pi}{2}$

Ans.: $r(\frac{\pi}{4}) = \langle 1 + \frac{\pi}{2} , \pi - 1 , e^{\frac{3\pi}{4}} - \frac{3\pi}{2} - 1 \rangle$

$v(t) = \int_0^t \langle -4\sin 2u , -4\cos 2u , 9e^{3u} \rangle + \langle 2, 0, 3 \rangle$

$= \langle 2\cos(2t) , -2\sin(2x) , 3e^{3t} \rangle + \langle 2, 0, 3 \rangle - \langle 0, -4, 9 \rangle$

$= \langle 2\cos 2t + 2 , -2\sin(2t) + 4 , 3e^{3t} - 6 \rangle$

$r(t) = \int_0^t \langle 2\cos 2t + 2 , -2\sin(2t) + 4 , 3e^{3t} - 6 \rangle + \langle 0, 1, 1 \rangle$

$= \langle \sin(2t) + 2t , \cos(2t) + 4t + 1 , e^{3t} - 6t + 1 \rangle - \langle 0, 2, 2 \rangle$

$r(t) = \langle \sin(2t) + 2t , \cos(2t) + 4t - 1 , e^{3t} - 6t - 1 \rangle$

$r(\pi/4) = \langle 1 + \frac{\pi}{2} , \pi - 1 , e^{3\pi/4} - \frac{3\pi}{2} - 1 \rangle$

8. (12 points) Compute the (scalar-function) line-integral

$$\int_C (x + y + 2\,z)\, ds \longrightarrow \text{NOT } F \cdot dr! \text{ WATCH OUT!}$$

where the curve C is given by the parametric equation:

$$\mathbf{r}(t) = \langle\, t, 2t, 2t \rangle \quad , \quad 0 \le t \le 1 \quad .$$

Ans.: $\frac{21}{2}$

$$\int_C (x+y+2z)\,ds$$

$$r(t) = \langle \underbrace{t}_{x(t)}, \underbrace{2t}_{y(t)}, \underbrace{2t}_{z(t)} \rangle \qquad 0 \le t \le 1$$

$$r'(t) = \langle 1, 2, 2 \rangle$$

$$\|r'(t)\| = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$= 3\int_0^1 (t + 2t + 2(2t))\,dt$$

$$= 3\int_0^1 7t\,dt = 3\left(\frac{7t^2}{2}\right)\Big|_0^1$$

$$= \frac{21t^2}{2}\Big|_0^1 = \frac{21}{2}$$

9. (12 points)

$\nearrow$ If

$$\lim_{(x,y,z)\to(1,1,1)} f(x,y,z) = 1 \quad , \quad \lim_{(x,y,z)\to(1,1,1)} g(x,y,z) = 2$$

compute

$$\lim_{(x,y,z)\to(1,1,1)} \sin(\frac{\pi}{3} f(x,y,z)) \cos(\frac{\pi}{4} g(x,y,z))$$

60 $\frac{\sqrt{3}}{2}$ · 45°

Ans.: $\frac{\sqrt{6}}{8}$

$\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{}$

10. (12 points) Compute

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} \quad ,$$

where

$$\mathbf{F} = \langle x^2 + \sin(y+z)\,,\ y^2 + xz^3\,,\ z^2 + e^{xy}\,\rangle$$

and where S is the boundary (consisting of all six faces) of the cube

$$\{(x,y,z)\,|\,0 \leq x, y, z \leq 1\}$$

with the normal pointing **outward**.

Ans.:

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11) (12 points) By finding a function f such that $\mathbf{F} = \nabla f$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

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$$\mathbf{F}(x,y,z) = \langle 2e^{2x+3y+4z}\ ,\ 3e^{2x+3y+4z}\ ,\ 4e^{2x+3y+4z} \rangle \quad ,$$

$$C : x = t \quad , \quad y = 2t \quad , \quad z = t^2 \quad , \quad 0 \le t \le 1 \quad .$$

Ans: <

$x = t$ $\quad 0 \le t \le 1$ $\quad r(t) = \langle t, 2t, t^2 \rangle$

$y = 2t$

$z = t^2$ $\quad F = \langle 2e^{2x+3y+4z}, 3e^{2x+3y+4z}, 4e^{2x+3y+4z} \rangle$

$f_x = 2e^{2x+3y+4z}$

$f = e^{2x+3y+4z}$ $\quad f = \langle e^{2x+3y+4z}, e^{2x+3y+4z}, e^{2x+3y+4z} \rangle$

$f(t) = \langle e^{2t+6t+4t^2}, e^{2t+6t+4t^2}, e^{2t+6t+4t^2} \rangle$

$r(1) = \langle 1, 2, 1 \rangle$

$r(0) = \langle 0, 0, 0 \rangle$ $\rightarrow f(1,2,1) - f(0,0,0) =$

$\langle e^{18}, e^{18}, e^{18} \rangle - \langle 1, 1, 1 \rangle$

12. (12 points) Evaluate the line integral

$$\int_C 5y\,dx + 5x\,dy + 6z\,dz \quad ,$$

where $C : x = t^2\,,\, y = t\,,\, z = t^2\,,\, 0 \le t \le 1.$

Ans.: 8

$x = t^2 \rightarrow 2t \qquad 0 \le t \le 1$

$y = t \rightarrow 1$

$z = t^2 \rightarrow 2t$

$\int_0^1 5y(2t) + 5x(1) + 6z(2t)\,dt$

$= \int_0^1 10t^2 + 5t^2 + 6t^2(2t)\,dt$

$= \int_0^1 (15t^2 + 12t^3)\,dt$

$= 5t^3 + 3t^4 \Big|_0^1 = 5 + 3 = \boxed{8}$

13. (12 points) Evaluate

$$\iiint_E \frac{1}{\sqrt{x^2+y^2+z^2}}\, dV \quad ,$$

where E is the hemisphere

$$\{(x,y,z)\,|\,x^2+y^2+z^2 \leq 100, z<0\} \quad .$$

Ans.:

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14. (12 points) Evaluate the quadruple integral

$$\iiiint_E 360\,x\,dV \quad ,$$

where

$$E = \{(x,y,z,w)\,|\,0 \le w \le 1\,,\,0 \le z \le w\,,\,0 \le y \le z\,,\,0 \le x \le y\} \quad .$$

Ans.: 3

$\int_0^1 \int_0^w \int_0^z \int_0^y 360x\, dx\, dy\, dz\, dw$

$\int_0^y 360x\, dx$

$= 180x^2 \Big|_0^y = 180y^2$

$\int_0^z 180y^2\, dy = 60y^3 \Big|_0^z = 60z^3$

$\int_0^w 60z^3\, dz = 15z^4 \Big|_0^w = 15w^4$

$\int_0^1 15w^4\, dw = 3w^5 \Big|_0^1 = 3$

15. (12 points) Find the Jacobian of the transformation from (u, v)-space to (x, y)-space.

$$x = 3\sin(2u + v) \quad , \quad y = u + v + \cos(u + v) \quad ,$$

at the point $(u, v) = (0, 0)$.

Ans.: 3

$x = 3\sin(2u+v)$ @ $(0,0)$

$y = u+v+\cos(u+v)$

$$\begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} 6\cos(2u+v) & 3\cos(2u+v) \\ 1-\sin(u+v) & 1-\sin(u+v) \end{vmatrix}_{@(0,0)}$$

$$= \begin{vmatrix} 6 & 3 \\ 1-0 & 1-0 \end{vmatrix} = 6 - 3 = 3$$

16. (12 points) Find the local maximum and minimum **points** and saddle point(s) of the function $f(x, y) = x^3 + y^2 - 6xy$

Local maximum points(s): None

Local minimum points(s): (6, 18)

saddle point(s): (0,0)

$f(x,y) = x^3 + y^2 - 6xy$

$f_x = 3x^2 - 6y = 0$

$f_{xx} = 6x$

$f_y = 2y - 6x = 0$

$f_{yy} = 2$

$f_{yx} = -6$

$\longrightarrow$ $3(x^2 - 2y) = 0$

$2(y - 3x) = 0$

$3x^2 - 6\cancel{y} = 0$

$-18x + 6y = 0$

$= -18x + 3x^2 = 0$

$3x(x - 6) = 0$

$x = 0, 6$

$1 \rightarrow (0,0)$

$2 \rightarrow (6, 18)$

$2y - 36 = 0$

$D = f_{xx} f_{yy} - [f_{xy}]^2$

$D = 6x(2) - 36$

$D = 12x - 36$

12
6
72

$D_1 = -36 = -$ $D < 0$

$D_2 = 72 - 36 = +$ $D > 0$ $f_{xx} = > 0$ $(6, 18)$

17. (8 points) Use the Divergence Theorem to calculate the surface integral $\int\int_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = \langle \overset{P}{x + y}, \overset{Q}{y + z}, \overset{R}{x + z} \rangle \quad ,$$

where S is the sphere (center $(1, -2, 4)$ and radius 10), in other words the region in 3D space:

$$\{(x, y, z) \,|\, (x - 1)^2 + (y + 2)^2 + (z - 4)^2 = 100\} \quad .$$

Spherical coordinates

$$\iint_S F \cdot dS = \iiint_E \operatorname{div} F \, dV$$

$$\operatorname{div} F = 1 + 1 + 1 = 3$$

$$= 3 \iiint$$