13.3 Homework

(3) $r(t) = \langle 2t, \ln(t), t^2 \rangle, \quad 1 \le t \le 4$

$\rightarrow \int_a^b |r'(t)|\,dt$

$\rightarrow r'(t) = \langle 2, \frac{1}{t}, 2t \rangle$

$\rightarrow |r'(t)| = \sqrt{4 + \frac{1}{t^2} + 4t^2} = \sqrt{\frac{4t^2 + 1 + 4t^4}{t^2}} = \frac{2t+1}{t}$

$\rightarrow \int_1^4 \left(\frac{2t+1}{t}\right) dt = \int_1^4 \left(2t + \frac{1}{t}\right) dt = \left[t^2 + \ln(t)\right]\Big|_1^4$

$\rightarrow (16 + \ln(4)) - (1 + \cancel{\ln(1)}^{0}) = 15 + \ln(4)$

$\rightarrow \boxed{L = 15 + \ln(4)}$

---

(9) $r(t) = \langle t^2, 2t^2, t^3 \rangle, \quad a = 0$

$\rightarrow \int_0^t |r'(t)|\,dt$

$\rightarrow r'(t) = \langle 2t, 4t, 3t^2 \rangle$

$\rightarrow |r'(t)| = \sqrt{4t^2 + 16t^2 + 9t^4} = \sqrt{20t^2 + 9t^4} = \sqrt{t^2(20 + 9t^2)} = t\sqrt{20 + 9t^2}$

$\rightarrow \int_0^t \left(t\sqrt{20 + 9t^2}\right) dt \Big/ \begin{matrix} u = 20 + 9t^2 \\ du = 18t \end{matrix} \Big/ \frac{1}{18}\int \sqrt{u}\,du = \frac{1}{\cancel{18}\,9} \cdot \frac{\cancel{2}}{3} u^{3/2} = \frac{1}{27}\left[20 + 9t^2\right]^{3/2}\Big|_0^t$

$\rightarrow \frac{1}{27}\left[(20 + 9t^2)^{3/2} - (20)^{3/2}\right]$

$\rightarrow \boxed{s(t) = \frac{1}{27}\left[(20 + 9t^2)^{3/2} - (20)^{3/2}\right]}$

---

(11) $r(t) = \langle 2t + 3, 4t - 3, 5 - t \rangle, \quad t = 4$

$\rightarrow r'(t) = \langle 2, 4, -1 \rangle$

$\rightarrow |r'(t)| = \sqrt{4 + 16 + 1} = \sqrt{21}$

$\rightarrow$ $\boxed{\text{Speed at } t = 4 \text{ is } \sqrt{21}.}$

---

(13) $r(t) = \langle t, \ln(t), (\ln t)^2 \rangle, \quad t = 1$

$\rightarrow r'(t) = \langle 1, \frac{1}{t}, \frac{2}{t}\ln(t) \rangle$

$\rightarrow r'(1) = \langle 1, 1, 0 \rangle$

$\rightarrow \boxed{|r'(1)| = \sqrt{2}}$

---

15 $r(t) = \langle \sin 3t, \cos 4t, \cos 5t \rangle, \ t = \frac{\pi}{2}$

$\rightarrow r'(t) = \langle 3\sin(3t), \ 4\cos(4t), \ 5\cos(5t) \rangle$

$\rightarrow r'\left(\frac{\pi}{2}\right) = \langle 3\sin\left(\frac{3\pi}{2}\right), \ 4\cos(2\pi), 5\cos\left(\frac{5\pi}{2}\right) \rangle = \langle -3, 4, 0 \rangle$

$\rightarrow \boxed{\left|r'\left(\frac{\pi}{2}\right)\right| = 5}$

13.4 Homework

① $r(t) = \langle 4t^2, 9t \rangle$

$\rightarrow \boxed{r'(t) = \langle 8t, 9 \rangle}$

$\rightarrow T(t) = \frac{r'(t)}{|r'(t)|} = \frac{\langle 8t, 9 \rangle}{\sqrt{64t^2+81}}$

$\rightarrow \boxed{T(t) = \frac{\langle 8t, 9 \rangle}{\sqrt{64t^2+81}}}$

$\rightarrow T(1) = \frac{\langle 8, 9 \rangle}{\sqrt{145}} = \left\langle \frac{8}{\sqrt{145}}, \frac{9}{\sqrt{145}} \right\rangle$

$\rightarrow \boxed{T(1) = \left\langle \frac{8}{\sqrt{145}}, \frac{9}{\sqrt{145}} \right\rangle}$

---

⑤ $r(t) = \langle \cos(\pi t), \sin(\pi t), t \rangle$

$\rightarrow \boxed{r'(t) = \langle -\pi \sin(\pi t), \pi \cos(\pi t), 1 \rangle}$

$\rightarrow T(t) = \frac{r'(t)}{|r'(t)|}$

$\rightarrow \frac{\langle -\pi \sin(\pi t), \pi \cos(\pi t), 1 \rangle}{\sqrt{\pi^2 \sin^2(\pi t) + \pi^2 \cos^2(\pi t) + 1}} = \boxed{\frac{\langle -\pi \sin(\pi t), \pi \cos(\pi t), 1 \rangle}{\sqrt{\pi^2+1}}}$

$\rightarrow T(1) = \frac{1}{\sqrt{\pi^2+1}} \langle -\pi \sin(\pi), \pi \cos(\pi), 1 \rangle$

$\rightarrow T(1) = \frac{1}{\sqrt{\pi^2+1}} \langle 0, -\pi, 1 \rangle$

$\rightarrow \boxed{T(1) = \left\langle 0, \frac{-\pi}{\sqrt{\pi^2+1}}, \frac{1}{\sqrt{\pi^2+1}} \right\rangle}$

---

⑦ $r(t) = \langle 1, e^t, t \rangle$

$\rightarrow K(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$

$\rightarrow r'(t) = \langle 0, e^t, 1 \rangle$

$\rightarrow r''(t) = \langle 0, e^t, 0 \rangle$

$\rightarrow r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 0 & e^t & 1 \\ 0 & e^t & 0 \end{vmatrix} = (-e^t)i - (0)j + (0)k = -e^t i$

$\rightarrow |r'(t) \times r''(t)| = \sqrt{e^{2t}} = e^t$

$\rightarrow |r'(t)| = \sqrt{e^{2t}+1}$

$\rightarrow \boxed{K(t) = \frac{e^t}{(e^{2t}+1)^{3/2}}}$

---

(11) $r(t) = \langle \frac{1}{t}, \frac{1}{t^2}, t^2 \rangle$, $t=-1$

$\rightarrow K(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$

$\rightarrow r'(t) = \langle -\frac{1}{t^2}, -\frac{2}{t^3}, 2t \rangle$

$\rightarrow r''(t) = \langle \frac{2}{t^3}, \frac{6}{t^4}, 2 \rangle$

$\rightarrow r'(-1) = \langle -1, 2, -2 \rangle$

$\rightarrow r''(-1) = \langle -2, 6, 2 \rangle$

$\rightarrow r'(-1) \times r''(-1) = \begin{vmatrix} i & j & k \\ -1 & 2 & -2 \\ -2 & 6 & 2 \end{vmatrix} = (4+12)i - (-2-4)j + (-6+4)k = 16i + 6j - 2k$

$\rightarrow |\langle 16, 6, -2 \rangle| = \sqrt{296} = 2\sqrt{74}$

$\rightarrow |\langle -1, 2, -2 \rangle| = \sqrt{9} = 3$

$\rightarrow \boxed{K(-1) = \frac{2\sqrt{74}}{27}}$

---

(17) $y = t^4$, $t = 2$

$\rightarrow K(t) = \frac{|f''(t)|}{(1+f'(t)^2)^{3/2}}$

$\rightarrow y' = 4t^3$

$\rightarrow y'' = 12t^2$

$\rightarrow K(t) = \frac{|12t^2|}{(1+(4t^3)^2)^{3/2}} = \frac{|12t^2|}{(1+16t^6)^{3/2}}$

$\rightarrow \boxed{K(2) = \frac{48}{(1025)^{3/2}}}$

---

(21) Show that the tractrix $r(t) = \langle t - \tanh(t), \operatorname{sech}(t) \rangle$ has the curvature function $K(t) = \operatorname{sech}(t)$.

$\rightarrow K(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$

$\rightarrow r'(t) = \langle \tanh^2(t), -\operatorname{sech}(t)\tanh(t) \rangle$

$\rightarrow r''(t) = \langle 2\tanh(t)\operatorname{sech}^2(t), \operatorname{sech}(t)\,t\left(\tanh^2(t) - \operatorname{sech}^2(t)\right) \rangle$

$\rightarrow r'(t) \times r''(t) = \tanh^2(t)\operatorname{sech}(t)$

$\rightarrow K(t) = \frac{\cancel{\tanh^2(t)}\operatorname{sech}(t)}{(\tanh(t))^{\cancel{3}}} = \frac{\operatorname{sech}(t)}{\tanh(t)} = \frac{1}{\sinh(t)} = \operatorname{csch}(t)$

$\rightarrow \boxed{K(t) = \operatorname{csch}(t)}$

13.5 Homework

③ $r(t) = \langle t^3,\ 1-t,\ 4t^2 \rangle,\quad t=1$

$\rightarrow v(t) = r'(t) = \langle 3t^2, -1, 8t \rangle$

$\rightarrow \boxed{v(1) = \langle 3, -1, 8 \rangle}$

$\rightarrow a(t) = v'(t) = \langle 6t, 0, 8 \rangle$

$\rightarrow \boxed{a(1) = \langle 6, 0, 8 \rangle}$

$\rightarrow s(1) = |v(1)| = \sqrt{9+1+64}$

$\rightarrow \boxed{s(1) = \sqrt{74}}$

---

⑤ $r(\theta) = \langle \sin\theta,\ \cos\theta,\ \cos 3\theta \rangle,\quad \theta = \frac{\pi}{3}$

$\rightarrow v(\theta) = r'(\theta) = \langle \cos\theta, -\sin\theta, -3\sin 3\theta \rangle$

$\rightarrow \boxed{v\left(\frac{\pi}{3}\right) = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2}, 0 \right\rangle}$

$\rightarrow a(\theta) = v'(\theta) = \langle -\sin\theta, -\cos\theta, -9\cos 3\theta \rangle$

$\rightarrow \boxed{a\left(\frac{\pi}{3}\right) = \left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}, 9 \right\rangle}$

$\rightarrow s\left(\frac{\pi}{3}\right) = \left|v\left(\frac{\pi}{3}\right)\right| = \sqrt{\frac{1}{4}+\frac{3}{4}+0} = \sqrt{1} = 1$

$\rightarrow \boxed{s\left(\frac{\pi}{3}\right) = 1}$

---

⑮ $a(t) = \langle t, 4 \rangle,\quad v(0) = \langle 3, -2 \rangle,\quad r(0) = \langle 0, 0 \rangle$

$\rightarrow v(t) = \int a(t)\,dt = \frac{t^2}{2}i + 4tj + 3i - 2j$

$\rightarrow \boxed{v(t) = \left\langle \frac{t^2}{2} + 3, 4t - 2 \right\rangle}$

$\rightarrow r(t) = \int v(t)\,dt = \left(\frac{t^3}{6} + 3t\right)i + \left(2t^2 - 2t\right)j$

$\rightarrow \boxed{r(t) = \left\langle \frac{t^3}{6} + 3t,\ 2t^2 - 2t \right\rangle}$

---

⑰ $a(t) = tK,\quad v(0) = i,\quad r(0) = j$

$\rightarrow v(t) = \int a(t)\,dt = \frac{t^2}{2}K + i$

$\rightarrow \boxed{v(t) = i + \frac{t^2}{2}K}$

$\rightarrow r(t) = \int v(t)dt = ti + \frac{t^3}{6}k + j$

$\rightarrow \boxed{r(t) = ti + j + \frac{t^3}{6}k}$

---

31) At a certain amount, a particle moving along a path has velocity $v = \langle 12, 20, 20 \rangle$ and acceleration $a = \langle 2, 1, -3 \rangle$. Is the particle speeding up or slowing down?

$\rightarrow 2(a \cdot v) = 2\langle 2, 1, -3 \rangle \cdot \langle 12, 20, 20 \rangle$

$\rightarrow -32$ which is less than 0

$\rightarrow$ $\boxed{\text{The speed is decreasing.}}$

---

33) $r(t) = \langle t, \cos t, \sin t \rangle$

$\rightarrow a_T = a \cdot T = \frac{a \cdot v}{||v||}, \quad a_N = a \cdot N = \sqrt{||a||^2 - |a_T|^2}$

$\rightarrow v(t) = r'(t) = \langle 1, -\sin(t), \cos(t) \rangle$

$\rightarrow a(t) = v'(t) = \langle 0, -\cos(t), -\sin(t) \rangle$

$\rightarrow a \cdot v = \langle 0, -\cos(t), -\sin(t) \rangle \cdot \langle 1, -\sin(t), \cos(t) \rangle$

$\rightarrow a \cdot v = 0 + \sin t \cos t - \sin t \cos t = 0$

$\rightarrow ||v|| = \sqrt{1^2 + \sin^2(t) + \cos^2(t)} = \sqrt{2}$

$\rightarrow \boxed{a_T = 0}$

$\rightarrow ||a||^2 = \left(\sqrt{0 + \cos^2 t + \sin^2 t}\right)^2 = 1$

$\rightarrow a_N = \sqrt{1 - 0} = 1$

$\rightarrow \boxed{a_N = 1}$

14.1 Homework

① $f(x,y) = x + yx^3, \quad (2,2), \quad (-1,4)$

$\rightarrow f(2,2) = 2 + 2 \cdot 2^3$

$\rightarrow \boxed{f(2,2) = 18}$

$\rightarrow f(-1,4) = -1 + 4 \cdot (-1)^3$

$\rightarrow \boxed{f(-1,4) = -5}$

---

③ $h(x,y,z) = xyz^{-2}, \quad (3,8,2), \quad (3,-2,-6)$

$\rightarrow h(3,8,2) = 3 \cdot 8 \cdot \frac{1}{2^2}$

$\rightarrow \boxed{h(3,8,2) = 6}$

$\rightarrow h(3,-2,-6) = 3 \cdot (-2) \cdot \frac{1}{(-6)^2}$

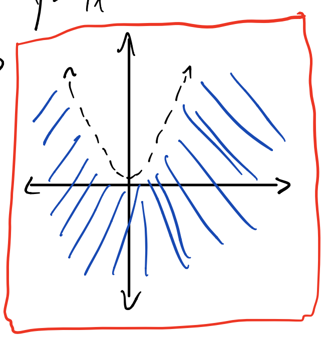
$\rightarrow \boxed{h(3,-2,-6) = \frac{1}{6}}$

---

⑦ $f(x,y) = \ln(4x^2 - y)$

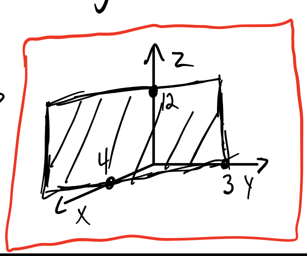
$\rightarrow y = 4x^2$

$\rightarrow$



㉑ $f(x,y) = 12 - 3x - 4y$

$\rightarrow$

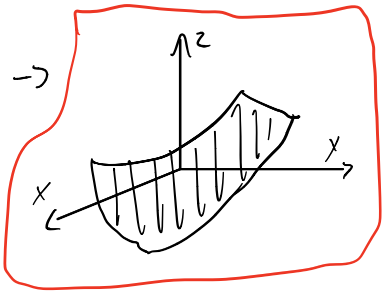


$\rightarrow$ Horizontal trace: $3x + 4y = 12 - c$ in plane $z = c$

$\rightarrow$ Vertical trace: $z = (12 - 3a) - 4y$ and $z = -3x + (12 - 4a)$ in $x = a$ and $y = a$

---

(23) $f(x,y) = x^2+4y^2$

→ 

→ The horizontal traces are ellipses for $c>0$

→ Vertical trace in plane $x=a$ is parabola $z=a^2+4y^2$

→ Vertical trace in place $y=a$ is parabola $z=x^2+4a^2$

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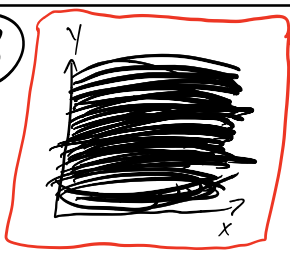
(33) 

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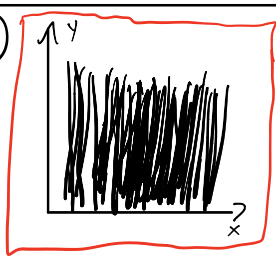
(35) 

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(37) $f(x,y) = qx + ry + s = c$

→ $f(0,0) = s = 6$

→ $f(-3,0) = q = 2$

→ $f(0,-1) = r = 6$

→ $\boxed{m=6, \ 2x+6y+6}$

→ $f(0,0) = s = 3$

→ $f(-3,0) = q = 1$

→ $f(0,-1) = r = 3$

→ $\boxed{m=3, \ x+3y+3}$

14.2 Homework

(5) $\lim_{(x,y)\to(\frac{\pi}{4},0)} \tan x \cos y$

$\to \lim_{(x,y)\to(\frac{\pi}{4},0)} \tan\left(\frac{\pi}{4}\right)\cos(0) = 1 \cdot 1$

$\to \boxed{1}$

---

(15) Let $f(x,y) = \frac{x^3+y^3}{xy^2}$. Set $y = mx$ and show that the resulting limit depends on m, and therefore the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

$\to y = mx$

$\to \lim_{x\to 0} \left(\frac{\cancel{x^3}+m^3\cancel{x^3}}{\cancel{x^3}m^2}\right) = \lim_{x\to 0}\left(\frac{1+m^3}{m^2}\right) = \frac{1+m^3}{m^2}$

$\to$ $\boxed{\text{This limit does not exist because it depends on } m.}$

---

(21) $\lim_{(x,y)\to(0,0)} \left(\frac{xy}{3x^2+2y^2}\right)$

$\to y = cx$

$\to \lim_{x\to 0} \left(\frac{cx^2}{3x^2+2c^2x^2}\right) = \lim_{x\to 0}\left(\frac{c}{3+2c^2}\right) = \frac{c}{3+2c^2}$

$\to$ $\boxed{\text{This limit does not exist because it depends on } c.}$

---

(23) $\lim_{(x,y,z)\to(0,0,0)} \left(\frac{x+y+z}{x^2+y^2+z^2}\right)$

$\to \lim_{x\to 0} \left(\frac{x}{x^2}\right) = \lim_{x\to 0}\left(\frac{1}{x}\right) = \frac{1}{0}$ X

$\rightarrow$ $\boxed{\text{This limit does not exist.}}$

---

(27) $$\lim_{(z,w)\to(-2,1)} \left(\frac{z^4\cos(\pi w)}{e^{2+w}}\right)$$

$$\rightarrow \lim_{(z,w)\to(-2,1)} \left(\frac{16\cos(\pi)}{e^{-1}}\right) = \boxed{-16e}$$

---

(31) $$\lim_{(x,y)\to(3,4)} \left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

$$\rightarrow \lim_{(x,y)\to(3,4)} \left(\frac{1}{\sqrt{9+16}}\right) = \boxed{\frac{1}{5}}$$

---

(35) $$\lim_{(x,y)\to(-3,-2)} \left(x^2y^3+4xy\right)$$

$$\rightarrow \lim_{(x,y)\to(-3,-2)} \left(9\cdot(-8)+4\cdot(-3)\cdot(-2)\right) = \boxed{-48}$$
