# 13.3 - 13.5, 14.1, 14.2 HW

9/27/20

## 13.3

3. $r(t) = \langle 2t, \ln t, t^2 \rangle, \quad 1 \le t \le 4$

$r'(t) = \langle 2, 1/t, t^2 \rangle \Rightarrow S = \int_1^4 \|r'(t)\| \, dt = \int_1^4 \sqrt{2^2 + \left(\frac{1}{t}\right)^2 + (2t)^2} \, dt$

$= \int_1^4 \sqrt{4t^2 + 4 + \frac{1}{t^2}} \, dt = \int_1^4 \sqrt{\left(2t + \frac{1}{t}\right)^2} \, dt = \int_1^4 \left(2t + \frac{1}{t}\right) dt = \ln t \Big|_1^4$

$= (16 + \ln 4) - (1 + \ln 1) = 15 + \ln 4$

9. $r(t) = \langle t^2, 2t^2, t^3 \rangle, \quad a = 0$

$r'(t) = \langle 2t, 4t, 3t^2 \rangle \Rightarrow \|r'(t)\| = \sqrt{(2t)^2 + (4t)^2 + (3t^2)^2}$

$= \sqrt{4t^2 + 16t^2 + 9t^4} = \sqrt{20 + 9t^2}\, t$

$s(t) = \int_0^t \|r'(u)\| \, du = \int_0^t \sqrt{20 + 9u^2}\, u \, du$ Sub: $v = 20 + 9u^2, \; dv = 18u\,du$

$s(t) = \frac{1}{18} \int_{20}^{20+9t^2} v^{1/2} \, dv = \frac{1}{18} \cdot \frac{2}{3} v^{3/2} \Big|_{20}^{20+9t^2} = \frac{1}{27}\left((20 + 9t^2)^{3/2} - 20^{3/2}\right)$

11. $r(t) = \langle 2t+3, 4t-3, 5-t \rangle, \quad t = 4 \Rightarrow r'(t) = \langle 2, 4, -1 \rangle$

$v(t) = \|r'(t)\| = \sqrt{2^2 + 4^2 + (-1)^2} = \sqrt{21} \approx 4.58$

13. $r(t) = \langle t, \ln t, (\ln t)^2 \rangle, \quad t = 1 \Rightarrow r'(t) = \langle 1, 1/t, 2\ln t / t \rangle$

$v(t) = \sqrt{1^2 + \left(\frac{1}{t}\right)^2 + \left(\frac{2\ln t}{t}\right)^2} = \frac{1}{|t|}\sqrt{t^2 + 1 + 4\ln^2 t}$

At $t = 1$, $v(1) = 1/1 \sqrt{1 + 1 + 4\ln^2 1} = \sqrt{2}$

15. $r(t) = \langle \sin 3t, \cos 4t, \cos 5t \rangle, \quad t = \pi/2 \Rightarrow r'(t) = \langle 3\cos 3t, -4\sin 4t, -5\sin 5t \rangle$

$v(\pi/2) = \|\langle 0, 0, -5 \rangle\| = 5$ $\quad r'(\pi/2) = \langle 3\cos 3\pi/2, -4\sin 2\pi, -5\sin 5\pi/2 \rangle$

$r'(\pi/2) = \langle 0, 0, -5 \rangle$

## 13.4

1. $r(t) = \langle 4t^2, 9t \rangle$

$r'(t) = \langle 8t, 9 \rangle \Rightarrow \|r'(t)\| = \sqrt{(8t)^2 + 9^2} = \sqrt{64t^2 + 81}$

$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{1}{\sqrt{64t^2 + 81}} \langle 8t, 9 \rangle$

$T(1) = \frac{1}{\sqrt{64 + 81}} \langle 8, 9 \rangle = \left\langle \frac{8}{\sqrt{145}}, \frac{9}{\sqrt{145}} \right\rangle$

5. $r(t) = \langle \cos \pi t, \sin \pi t, t \rangle$

$r'(t) = \langle -\pi \sin \pi t, \pi \cos \pi t, 1 \rangle \Rightarrow \|r'(t)\| = \sqrt{(-\pi \sin \pi t)^2 + (\pi \cos \pi t)^2 + 1^2}$

$= \sqrt{\pi^2(\sin^2 \pi t + \cos^2 \pi t) + 1} = \sqrt{\pi^2 + 1}$

$T(t) = \dfrac{r'(t)}{\|r'(t)\|} = \dfrac{1}{\sqrt{\pi^2+1}} \langle -\pi \sin \pi t, \pi \cos \pi t, 1 \rangle$

$T(1) = \dfrac{1}{\sqrt{\pi^2+1}} \langle -\pi \sin \pi, \pi \cos \pi, 1 \rangle = \dfrac{1}{\sqrt{\pi^2+1}} \langle 0, -\pi, 1 \rangle = \left\langle 0, -\dfrac{\pi}{\sqrt{\pi^2+1}}, \dfrac{1}{\sqrt{\pi^2+1}} \right\rangle$

7. $r(t) = \langle 1, e^t, t \rangle \Rightarrow r'(t) = \langle 0, e^t, 1 \rangle$, $r''(t) = \langle 0, e^t, 0 \rangle$

$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 0 & e^t & 1 \\ 0 & e^t & 0 \end{vmatrix} = \begin{vmatrix} e^t & 1 \\ e^t & 0 \end{vmatrix} i - \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} j + \begin{vmatrix} 0 & e^t \\ 0 & e^t \end{vmatrix} K = -e^t i = \langle -e^t, 0, 0 \rangle$

$\|r'(t) \times r''(t)\| = |\langle -e^t, 0, 0 \rangle| = e^t$

$\|r'(t)\| = \sqrt{0^2 + (e^t)^2 + 1^2} = \sqrt{1 + e^{2t}}$

$\kappa(t) = \dfrac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} = \dfrac{e^t}{(\sqrt{1+e^{2t}})^3} = \dfrac{e^t}{(1+e^{2t})^{3/2}}$

11. $r(t) = \langle 1/t, 1/t^2, t^2 \rangle$, $t = -1$

$r'(t) = \langle -t^{-2}, -2t^{-3}, 2t \rangle \Rightarrow r'(-1) = \langle -1, 2, -2 \rangle$

$r''(t) = \langle 2t^{-3}, 6t^{-4}, 2 \rangle \Rightarrow r''(-1) = \langle -2, 6, 2 \rangle$

$r'(-1) \times r''(-1) = \langle 16, 6, -2 \rangle$

$\|r'(-1)\| = \sqrt{(-1)^2 + 2^2 + (-2)^2} = 3$

$\|r'(-1) \times r''(-1)\| = \sqrt{16^2 + 6^2 + (-2)^2} = \sqrt{296} = 2\sqrt{74}$

$\kappa(-1) = \dfrac{\|r'(-1) \times r''(-1)\|}{\|r'(-1)\|^3} = \dfrac{2\sqrt{74}}{3^3} = \dfrac{2\sqrt{74}}{27}$

17. $y = t^4$, $t = 2 \Rightarrow f(t) = t^4$, $f'(t) = 4t^3$, $f''(t) = 12t^2$

$\kappa(t) = \dfrac{|f''(t)|}{(1 + f'(t)^2)^{3/2}} = \dfrac{12t^2}{(1 + (4t^3)^2)^{3/2}} = \dfrac{12t^2}{(1 + 16t^6)^{3/2}}$

$\kappa(2) = \dfrac{12 \cdot 2^2}{(1 + 16 \cdot 2^6)^{3/2}} = \dfrac{48}{(1{,}025)^{3/2}} \approx 0.0015$

21. $r(t) = \langle x(t), y(t) \rangle \Rightarrow x(t) = t - \tanh t$, $y(t) = \operatorname{sech} t$

$x'(t) = 1 - \operatorname{sech}^2 t = \tanh^2 t$

$x''(t) = -2 \operatorname{sech} t (-\operatorname{sech} t \tanh t) = 2 \operatorname{sech}^2 t \tanh t$

$y'(t) = -\operatorname{sech} t \tanh t$

$y''(t) = -(-\operatorname{sech} t \tanh^2 t + \operatorname{sech}^3 t) = \operatorname{sech} t (\tanh^2 t - \operatorname{sech}^2 t) = \operatorname{sech} t (1 - 2\operatorname{sech}^2 t)$

# 13.3 - 13.5, 14.1, 14.2 HW Cont.

$2\operatorname{sech}^3 t \tanh^2 t$

$\|r' \times r''\|$: $x'(t)y''(t) - x''(t)y'(t) = \tanh^2 t \operatorname{sech} t(1 - 2\operatorname{sech}^2 t) +$

$= \tanh^2 t\left[\operatorname{sech} t - 2\operatorname{sech}^3 t + 2\operatorname{sech}^3 t\right] = \tanh^2 t \operatorname{sech} t$

$r'$: $x'(t)^2 + y'(t)^2 = \tanh^4 t + \operatorname{sech}^2 t \tanh^2 t = \tanh^2 t(\tanh^2 t + \operatorname{sech}^2 t)$

$= \tanh^2 t$

$\|r'\|^3 = (\tanh^2 t)^{3/2} = \tanh^3 t$

$$\kappa(t) = \frac{|\operatorname{sech} t \tanh^2 t|}{\tanh^3 t} = \frac{\operatorname{sech} t \tanh^2 t}{\tanh^3 t} = \frac{\operatorname{sech} t}{\tanh t}$$

## 13.5

3. $r(t) = \langle t^3, 1-t, 4t^2\rangle$, $t = 1$

$v(t) = r'(t) = \langle 3t^2, -1, 8t\rangle \Rightarrow v(1) = \langle 3, -1, 8\rangle$

$a(t) = r''(t) = \langle 6t, 0, 8\rangle \Rightarrow a(1) = \langle 6, 0, 8\rangle$

Speed: $v(1) = \|v(1)\| = \sqrt{3^2 + (-1)^2 + 8^2} = \sqrt{74}$

5. $r(\theta) = \langle \sin\theta, \cos\theta, \cos 3\theta\rangle$, $\theta = \pi/3$

$v(\theta) = r'(\theta) = \langle \cos\theta, -\sin\theta, -3\sin 3\theta\rangle$

$\Rightarrow v(\pi/3) = \langle \cos \pi/3, -\sin \pi/3, -3\sin\pi\rangle = \langle 1/2, -\sqrt{3}/2, 0\rangle$

$a(\theta) = r''(\theta) = \langle -\sin\theta, -\cos\theta, -9\cos 3\theta\rangle$

$\Rightarrow a(\pi/3) = \langle -\sin\pi/3, -\cos\pi/3, -9\cos\pi\rangle = \langle -\sqrt{3}/2, -1/2, 9\rangle$

Speed: $v(\pi/3) = \|v(\pi/3)\| = \sqrt{(1/2)^2 + (-\sqrt{3}/2)^2 + 0^2} = 1$

8. $a(t) = \langle t, 4\rangle$, $v(0) = \langle 3, -2\rangle$, $r(0) = \langle 0, 0\rangle$

$$v(t) = \int_0^t \langle u, 4\rangle\, du = \left\langle \frac{u^2}{2}, 4u\right\rangle\Big|_0^t + v_0 = \left\langle \frac{t^2}{2}, 4u\right\rangle + v_0$$

$v(0) = \langle 0, 0\rangle + v_0 = \langle 3, -2\rangle \Rightarrow v_0 = \langle 3, -2\rangle$

$v(t) = \langle t^2/2, 4t\rangle + \langle 3, -2\rangle = \langle t^2/2 + 3, 4t - 2\rangle$

$$r(t) = \int_0^t \left\langle \frac{u^2}{2} + 3, 4u - 2\right\rangle du = \left\langle \frac{u^3}{6} + 3u, 2u^2 - 2u\right\rangle\Big|_0^t + r_0$$

$$= \left\langle \frac{t^3}{6} + 3t, 2t^2 - 2t\right\rangle + r_0$$

$r(0) = \langle 0, 0\rangle + r_0 = \langle 0, 0\rangle \Rightarrow r_0 = \langle 0, 0\rangle$

$$r(t) = \left\langle \frac{t^3}{6} + 3t, 2t^2 - 2t\right\rangle$$

/20

17. $a(t) = tK$, $v(0) = i$, $r(0) = j$

$v(t) = \int_0^t uK\,du = \frac{u^2}{2}K \Big|_0^t + v_0 = \frac{t^2}{2}K + v_0$

$v(0) = 0^2/_2\, K + v_0 = i \Rightarrow v_0 = i$

$v(t) = t^2/_2\, K + i = i + t^2/_2\, K$

$r(t) = \int_0^t \left(i + \frac{u^2}{2}K\right)du = ui + \frac{u^3}{6}K \Big|_0^t + r_0 = ti + \frac{t^3}{6}K + r_0$

$r(0) = 0i + 0K + r_0 = j \Rightarrow r_0 = j$

$r(t) = ti + j + t^3/6\, K$

Particle speeding up or slowing down? $\Rightarrow$ $\|v\|$ or $\|v\|^2$ incr. or decr. ?

$(\|v\|^2)' = (v \cdot v)' = 2v' \cdot v = 2 \cdot a \cdot v = 2\langle 2, 1, -3\rangle \cdot \langle 12, 20, 20\rangle$

$= 2 \cdot \langle 24 + 20 - 60\rangle = -32 < 0 \Rightarrow$ Speed decr.

$r(t) = \langle t, \cos t, \sin t\rangle$

$a_T = a \cdot T$, $a_N = \frac{\|a \times v\|}{\|v\|}$

$v(t) = r'(t) = \langle 1, -\sin t, \cos t\rangle \Rightarrow \|v(t)\| = \sqrt{1 + (-\sin t)^2 + \cos^2 t} = \sqrt{2}$

$a(t) = r''(t) = \langle 0, -\cos t, -\sin t\rangle$

$T(t) = \frac{v(t)}{\|v(t)\|} = \frac{1}{\sqrt{2}}\langle 1, -\sin t, \cos t\rangle$

Since speed is constant ($v = \|v(t)\| = \sqrt{2}$), the tangential component of acc. is 0 ($a_T = 0$)

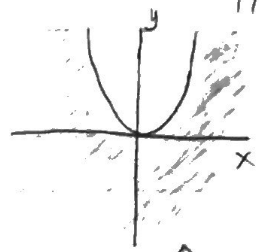
$a \times v = \begin{vmatrix} i & j & K \\ 0 & -\cos t & -\sin t \\ 1 & -\sin t & \cos t \end{vmatrix} = \begin{vmatrix} -\cos t & -\sin t \\ -\sin t & \cos t \end{vmatrix} i - \begin{vmatrix} 0 & -\sin t \\ 1 & \cos t \end{vmatrix} j + \begin{vmatrix} 0 & -\cos t \\ 1 & -\sin t \end{vmatrix} K$

$= -(\cos^2 t + \sin^2 t)i - \sin t\, j + \cos t\, K = -i - \sin t\, j + \cos t\, K = \langle -1, -\sin t, \cos t\rangle$

$a_N = \frac{\|a \times v\|}{\|v\|} = \frac{\sqrt{(-1)^2 + (-\sin t)^2 + \cos^2 t}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} = 1$

# 13.3-13.5, 14.1, 14.2 HW Cont.

9/24/20

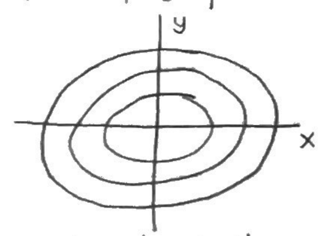



14.1

1. $f(x,y) = x + yx^3$, $(2,2)$, $(-1,4)$
$f(2,2) = 2 + 2(2)^3 = 2 + 16 = 18$
$f(-1,4) = -1 + 4(-1)^3 = -1 - 4 = -5$

3. $h(x,y,z) = xyz^{-2}$, $(3,8,2)$, $(3,-2,-6)$
$h(3,8,2) = 3(8)(2)^{-2} = 24/4 = 6$
$h(3,-2,-6) = 3(-2)(-6)^{-2} = -6/36 = -1/6$

7. $f(x,y) = \ln(4x^2 - y)$ => defined if $4x^2 - y > 0$ => $y < 4x^2$. The domain is the region in the xy-plane that is below the parabola $y = 4x^2$

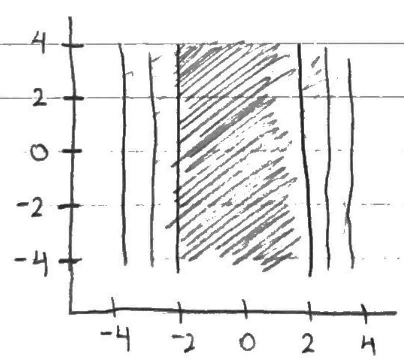



21. $f(x,y) = 12 - 3x - 4y$ => USE MAPLE TO GRAPH

23. $f(x,y) = x^2 + 4y^2$ => USE MAPLE TO GRAPH

33. $f(x,y) = x^2 + 4y^2$
The level curves are $x^2 + 4y^2 = c$. These are ellipses centered at the origin in the xy-plane (contour map).

35. $f(x,y) = x^2$
The level curves are $x^2 = c$. For $c > 0$ these are 2 vert. lines $x = \sqrt{c}$ & $x = -\sqrt{c}$ & for $c = 0$ it is the y-axis. Draw a contour map using contour interval $m = 4$ & $c = 0, 4, 8, 12, 16, 20$:




37. OPTIONAL

14.2

In Exs. 9-12, assume that $\lim_{(x,y)\to(2,5)} f(x,y) = 3$ & $\lim_{(x,y)\to(2,5)} g(x,y) = 7$

9. $\lim_{(x,y)\to(2,5)} (g(x,y) - 2f(x,y)) = 7 - 2(3) = 7 - 6 = 1$

7/2

11. $\lim_{(x,y)\to(2,5)} e^{f(x,y)^2 - g(x,y)} = e^{3^2-7} = e^{9-7} = e^2$

$f(x, mx) = \dfrac{x^3 + (mx)^3}{x(mx)^2} = \dfrac{(m^3+1)x^3}{m^2x^3} = \dfrac{m^3+1}{m^2}$

$y = mx$ ?

$m=2$ : $\lim_{x\to 0} f(x, mx) = \lim_{x\to 0} (2^3+1)/(2^2) = (8+1)/(4) = 9/4$

$m=1$ : $\lim_{x\to 0} f(x, mx) = \lim_{x\to 0} (1^3+1)/(1^2) = (1+1)/(1) = 2/1 = 2$

Therefore $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE

$\lim_{(x,y)\to(0,0)} \dfrac{xy}{3x^2+2y^2} = \text{DNE}$

$\lim_{x\to 0} \dfrac{x(mx)}{3x^2+2(mx)^2} = \lim_{x\to 0} \dfrac{mx^2}{x^2(3+2m^2)} = \lim_{x\to 0} \dfrac{m}{3+2m^2}$ => Diff. for all vals. of m

$\lim_{(x,y,z)\to(0,0,0)} \dfrac{x+y+z}{x^2+y^2+z^2} = \text{DNE}$ $y=z=0$

$\lim_{x\to 0} \dfrac{x+0+0}{x^2+0^2+0^2} = \lim_{x\to 0} \dfrac{x}{x^2} = \lim_{x\to 0} \dfrac{1}{x} = \infty$

$\lim_{(z,w)\to(-2,1)} \dfrac{z^4\cos(\pi w)}{e^{z+w}} = \dfrac{(-2)^4\cos(\pi)}{e^{-2+1}} = \dfrac{16(-1)}{e^{-1}} = -16e$

31. $\lim_{(x,y)\to(3,4)} \dfrac{1}{\sqrt{x^2+y^2}} = \dfrac{1}{\sqrt{3^2+4^2}} = \dfrac{1}{\sqrt{9+16}} = \dfrac{1}{\sqrt{25}} = \dfrac{1}{5}$

35. $\lim_{(x,y)\to(-3,-2)} (x^2y^3+4xy) = 9(-8)+4(-3)(-2) = -72+24 = -48$