Chs. 12.3, 12.4, 12.5, 13.1, 13.2 HW 9/20/20

12.3

1. $\langle 1,2,1\rangle \cdot \langle 4,3,5\rangle = 4+6+5 = 15$

13. $\langle 1,1,1\rangle, \langle 1,-2,-2\rangle \Rightarrow \langle 1,1,1\rangle \cdot \langle 1,-2,-2\rangle = 1-2-2 = -3$

Dot product is negative $\Rightarrow \angle$ between vectors is obtuse

21. $i+j, \quad j+2k \Rightarrow$ Let: $v = i+j$ & $w = j+2k$

$\|v\| = \|i+j\| = \sqrt{1^2+1^2} = \sqrt{2}$

$\|w\| = \|j+2k\| = \sqrt{1^2+2^2} = \sqrt{5}$

$v \cdot w = (i+j)\cdot(j+2k) = i\cdot j + 2i\cdot k + j\cdot j + 2j\cdot k = \|j\|^2 = 1$

$\cos\theta = v\cdot w / \|v\|\|w\| = 1/(\sqrt{2}\cdot\sqrt{5}) = 1/\sqrt{10}$

29. (a) $\langle b,3,2\rangle, \langle 1,b,1\rangle$; Vectors are orthogonal if scalar product $=0$

$\langle b,3,2\rangle \cdot \langle 1,b,1\rangle = 0$

$b\cdot 1 + 3\cdot b + 2\cdot 1 = 0$

$4b+2 = 0 \Rightarrow b = -1/2$

(b) $\langle 4,-2,7\rangle, \langle b^2,b,0\rangle$

$\langle 4,-2,7\rangle \cdot \langle b^2,b,0\rangle = 0$

$4b^2 - 2b = 0$

$2b(2b-1) = 0 \Rightarrow b=0$ or $b = 1/2$

31. Find 2 vectors that are not multiples of each other & are both orthogonal to $\langle 2,0,-3\rangle$

$\langle a,b,c\rangle \cdot \langle 2,0,-3\rangle = 0$

$2a - 3c = 0 \Rightarrow a = 3/2\, c \Rightarrow \langle 3/2\, c, b, c\rangle$

$c=0, b=1 \Rightarrow v_1 = \langle 0,1,0\rangle$

$c=2, b=2 \Rightarrow v_2 = \langle 3,2,2\rangle$

[illegible]. $u = 5i+7j-4k, \quad v = k$

The projection of u along v is: $u_{\parallel v} = \left(\frac{u\cdot v}{v\cdot v}\right)v = \frac{-4}{1}k = -4k$

$u\cdot v = (5i+7j-4k)\cdot k = -4k\cdot k = -4$

$v\cdot v = \|v\|^2 = \|k\|^2 = 1$

[illegible]. Find the length of $\overline{OP}$ in Figure 15. $\Rightarrow$ component of $u = \langle 3,5\rangle$ along $v = \langle 8,2\rangle$

$u\cdot v = \langle 3,5\rangle\cdot\langle 8,2\rangle = 34$

$v\cdot v = \|v\|^2 = 8^2+2^2 = 68$

$\Rightarrow \left\|\left(\frac{u\cdot v}{v\cdot v}\right)v\right\| = \frac{34}{68}\|v\| = \frac{34\sqrt{68}}{68}$

12.4

1. $\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = 1\cdot 3 - 2\cdot 4 = 3-8 = -5$

$\begin{vmatrix} 1 & 2 & 1 \\ 4 & -3 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} -3 & 0 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & -3 \\ 1 & 0 \end{vmatrix}$

$= 1\cdot(-3\cdot 1 - 0\cdot 0) - 2\cdot(4\cdot 1 - 0\cdot 1) + 1\cdot(4\cdot 0 - (-3)\cdot 1)$

$= -3 - 8 + 3 = -8$

$(i+j)\times k = i\times k + j\times k = -j + i$

$i\times k = -j$ & $j\times k = i$

$u\times v = \langle 1,1,0\rangle$, $u\times w = \langle 0,3,1\rangle$, $v\times w = \langle 2,-1,1\rangle$

$(u-2v)\times(u+2v) = (u-2v)\times u + (u-2v)\times 2v$

$= u\times u - 2v\times u + u\times 2v - 4v\times v = 0 + 2u\times v + 2u\times v - 0$

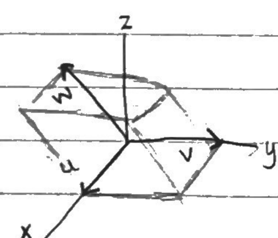
$= 0 + 4u\times v = \langle 4,4,0\rangle$

??? Let $v = \langle 3,0,0\rangle$ & $w = \langle 0,1,-1\rangle$. Determine $u = v\times w$ using the geometric properties of the cross product rather than form.

The cross product $u = v\times w$ is orthogonal to v.

Sketch & compute the volume of the parallelepiped spanned by $u = \langle 1,0,0\rangle$, $v = \langle 0,2,0\rangle$, $w = \langle 1,1,2\rangle$

$u\cdot(v\times w) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 1(4-0) - 0 + 0 = 4$



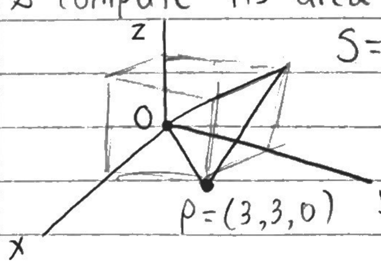
Calc. area of the ▱ spanned by $u = \langle 1,0,3\rangle$ & $v = \langle 2,1,1\rangle$

$u\times v = \begin{vmatrix} i & j & k \\ 1 & 0 & 3 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} i - \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} j + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} k = -3i - (1-6)j + k$

$= -3i + 5j + k$

$A = \|u\times v\| = \sqrt{(-3)^2 + 5^2 + 1^2} = \sqrt{35}$

Sketch △ w/ vertices at the origin O, $P = (3,3,0)$, & $Q = (0,3,3)$, & compute its area using cross products.



$S = \frac{1}{2}\|\vec{OP}\times\vec{OQ}\|$

$\vec{OP}\times\vec{OQ} = \begin{vmatrix} i & j & k \\ 3 & 3 & 0 \\ 0 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 3 & 3 \end{vmatrix} i - \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} j + \begin{vmatrix} 3 & 3 \\ 0 & 3 \end{vmatrix} k$

$= 9i - 9j + 9k = 9\langle 1,-1,1\rangle$

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$S = 1/2 \, \| 9 \langle 1,-1,1 \rangle \| = 9/2 \, \| \langle 1,-1,1 \rangle \| = 9/2 \sqrt{1^2 + (-1)^2 + 1^2} = 9\sqrt{3}/2$

45. Use cross products to find the area of the Δ in the xy-plane defined by $(1,2)$, $(3,4)$, & $(-2,2)$

Let: $P = (1,2,0)$, $Q = (3,4,0)$, $R = (-2,2,0)$

$T = 1/2 \, \| \vec{PQ} \times \vec{PR} \|$

$\vec{PQ} = \langle 3-1, 4-2, 0-0 \rangle = \langle 2,2,0 \rangle$

$\vec{PR} = \langle -2-1, 2-2, 0-0 \rangle = \langle -3,0,0 \rangle$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 2 & 2 & 0 \\ -3 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} i - \begin{vmatrix} 2 & 0 \\ -3 & 0 \end{vmatrix} j + \begin{vmatrix} 2 & 2 \\ -3 & 0 \end{vmatrix} k = 6k$$

$T = 1/2 \, \| 6k \| = 1/2 \cdot 6 = 3$

12.5

1. $n = \langle 1,3,2 \rangle$, $\langle 4,-1,1 \rangle$

$\langle 1,3,2 \rangle \cdot \langle x,y,z \rangle = \langle 1,3,2 \rangle \cdot \langle 4,-1,1 \rangle = 4 - 3 + 2 = 3$

$x + 3y + 2z = 3$

5. $n = i$, $(3,1,-9) \Rightarrow n = \langle 1,0,0 \rangle$

$\langle 1,0,0 \rangle \cdot \langle x,y,z \rangle = \langle 1,0,0 \rangle \cdot \langle 3,1,-9 \rangle = 3 + 0 + 0 = 3$

$x + 0 + 0 = 3 \Rightarrow x = 3$

9. Write the eq. of any plane through the origin.

$ax + by + cz = d$ that contains the point $(x,y,z) = (0,0,0)$

$\Rightarrow x + y + z = 0$

Which of the following statements are true of a plane that is parallel to the yz-plane?

(a) $n = \langle 0,0,1 \rangle$ is a normal vector $\Rightarrow \parallel$ to xy-plane FALSE

(b) $n = \langle 1,0,0 \rangle$ is a normal vector $\Rightarrow \parallel$ to yz-plane TRUE

(c) The eq. has the form $ay + bz = d$ FALSE

(d) The eq. has the form $x = d$. $\Rightarrow \langle 1,0,0 \rangle$ TRUE

13. $9x - 4y - 11z = 2 \Rightarrow n = \langle 9,-4,-11 \rangle$

15. $3(x-4) - 8(y-1) + 11z = 0 \Rightarrow n = \langle 3,-8,11 \rangle$

$P = (2,-1,4)$, $Q = (1,1,1)$, $R = (3,1,-2)$

$a = \vec{PQ} = \langle 1-2,\ 1-(-1),\ 1-4 \rangle = \langle -1, 2, -3 \rangle$

$b = \vec{PR} = \langle 3-2,\ 1-(-1),\ -2-4 \rangle = \langle 1, 2, -6 \rangle$

$n = a \times b = \begin{vmatrix} i & j & k \\ -1 & 2 & -3 \\ 1 & 2 & -6 \end{vmatrix} = \begin{vmatrix} 2 & -3 \\ 2 & -6 \end{vmatrix} i - \begin{vmatrix} -1 & -3 \\ 1 & -6 \end{vmatrix} j + \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} k$

$= -6i - 9j - 4k = \langle -6, -9, -4 \rangle$

Choose a point $\Rightarrow$ $Q = (1,1,1)$

$n \cdot \langle x, y, z \rangle = n \cdot \langle x_0, y_0, z_0 \rangle$

$\langle -6, -9, -4 \rangle \cdot \langle x, y, z \rangle = \langle -6, -9, -4 \rangle \cdot \langle 1, 1, 1 \rangle$

$-6x - 9y - 4z = -6 - 9 - 4 = -19$

$6x + 9y + 4z = 19$

☆. $P = (1,0,0)$, $Q = (0,1,1)$, $R = (2,0,1)$

$n \cdot \langle x, y, z \rangle = d$

$\vec{PQ} = \langle 0,1,1 \rangle - \langle 1,0,0 \rangle = \langle -1, 1, 1 \rangle$

$\vec{PR} = \langle 2,0,1 \rangle - \langle 1,0,0 \rangle = \langle 1, 0, 1 \rangle$

$n = \vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} i - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} j + \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} k$

$= i + 2j - k = \langle 1, 2, -1 \rangle$

$P = (1,0,0)$

$d = n \cdot \vec{OP} = \langle 1, 2, -1 \rangle \cdot \langle 1, 0, 0 \rangle = 1 \cdot 1 + 2 \cdot 0 + (-1) \cdot 0 = 1$

$\langle 1, 2, -1 \rangle \cdot \langle x, y, z \rangle = 1$

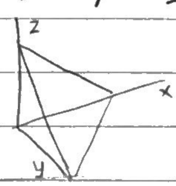
$x + 2y - z = 1$

??? ☆. Passes through $(-2,-3,5)$ & has normal vector $i + k$

$\langle 1, 0, 1 \rangle \cdot \langle x, y, z \rangle = \langle 1, 0, 1 \rangle \cdot \langle -2, -3, -5 \rangle = -2 + 0 - 5 = -7$

$\Rightarrow x + 0 + z = -7 \Rightarrow x + z = -7$

31. $x + y + z = 4$



??? ☆. Find all planes in $\mathbb{R}^3$ whose intersection w/ xz-plane is the line w/ eq. $3x + 2z = 5$

$ax + by + cz = d$, $y = 0$ $\Rightarrow$ $ax + cz = d$

$a = 3\lambda$, $c = 2\lambda$, $d = 5\lambda$

$(3\lambda)x + by + (2\lambda)z = 5\lambda$, $\lambda \neq 0$

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<u>13.1</u>

5. Find a vector parametrization of the line through $P=(3,-5,7)$ in the direction $v=\langle 3,0,1\rangle$.

$r(t) = \vec{OP} + tv = \langle 3,-5,7\rangle + t\langle 3,0,1\rangle = \langle 3+3t, -5, 7+t\rangle$
$= (3+3t)i - 5j + (7+t)K$

17. $r(t) = (9\cos t)i + (9\sin t)j \Rightarrow x(t) = 9\cos t, \; y(t) = 9\sin t$
$x^2+y^2 = 81\cos^2 t + 81\sin^2 t = 81(\cos^2 t + \sin^2 t) = 81$
This is the eq. of a ○ w/ $r=9$ centered at the origin. ○ lies in xy-plane.

<u>13.2</u> (#15 Optional)

3. $\lim\limits_{t\to 0}\left(e^{2t}i + \ln(t+1)j + 4K\right) = \left(\lim\limits_{t\to 0} e^{2t}\right)i + \left(\lim\limits_{t\to 0}\ln(t+1)\right)j + \left(\lim\limits_{t\to 0} 4\right)K$
$= e^0 i + (\ln 1)j + 4K = i + 4K$

5. Evaluate $\lim\limits_{h\to 0}\frac{r(t+h)-r(t)}{h}$ for $r(t) = \langle t^{-1}, \sin t, 4\rangle$

$= \frac{dr}{dt} = \left\langle \frac{d}{dt}(t^{-1}), \frac{d}{dt}(\sin t), \frac{d}{dt}(4)\right\rangle = \left\langle -\frac{1}{t^2}, \cos t, 0\right\rangle$

7. $r(t) = \langle t, t^2, t^3\rangle \Rightarrow \frac{dr}{dt} = \left\langle \frac{d}{dt}(t), \frac{d}{dt}(t^2), \frac{d}{dt}(t^3)\right\rangle = \langle 1, 2t, 3t^2\rangle$

9. $r(t) = \langle 1-t^2, 5t, 2t^3\rangle, \quad t=2$
Tangent line is parametrized by: $\ell(t) = r(2) + tr'(2)$
$r(2) = \langle 1-2^2, 5\cdot 2, 2\cdot 2^3\rangle = \langle -3, 10, 16\rangle$
$r'(t) = d/dt\,\langle 1-t^2, 5t, 2t^3\rangle = \langle -2t, 5, 6t^2\rangle \Rightarrow r'(2) = \langle -4, 5, 24\rangle$
$\ell(t) = \langle -3, 10, 16\rangle + t\langle -4, 5, 24\rangle = \langle -3-4t, 10+5t, 16+24t\rangle$

13. $r(s) = 4s^{-1}i - 8/3\, s^{-3}K, \quad s=2$
$\ell(s) = r(2) + sr'(2)$
$r(2) = 4(2)^{-1}i - 8/3\,(2)^{-3}K = 2i - 1/3\,K$
$r'(s) = d/ds\left(4s^{-1}i - 8/3\, s^{-3}K\right) = -4s^{-2}i + 8s^{-4}K \Rightarrow r'(2) = -i + 1/2\,K$
$\ell(t) = (2i - 1/3\,K) + s(-i + 1/2\,K) = (2-s)i + (1/2\,s - 1/3)K$

✓2

41. $$\int_{-2}^{2} (u^3 i + u^5 j)\,du = \left[\begin{array}{l} \displaystyle\int_{-2}^{2} u^3\,du = \frac{u^4}{4}\Big|_{-2}^{2} = \frac{16}{4} - \frac{16}{4} = 0 \\ \displaystyle\int_{-2}^{2} u^5\,du = \frac{u^6}{6}\Big|_{-2}^{2} = \frac{64}{6} - \frac{64}{6} = 0 \end{array}\right]$$

$$\int_{-2}^{2} (u^3 i + u^5 j)\,du = \left(\int_{-2}^{2} u^3\,du\right) i + \left(\int_{-2}^{2} u^5\,du\right) j = 0i + 0j$$

49. $r'(t) = t^2 i + 5tj + K \quad , \quad r(1) = j + 2K$

$$r(t) = \int (t^2 i + 5tj + K)\,dt = \left(\int t^2\,dt\right) i + \left(\int 5t\,dt\right) j + \left(\int 1\,dt\right) K$$

$$= \left(\tfrac{1}{3} t^3\right) i + \left(\tfrac{5}{2} t^2\right) j + tK + c$$

$$j + 2K = r(0) = \left(\tfrac{1}{3} \cdot 1^3\right) i + \left(\tfrac{5}{2} \cdot 1^2\right) j + 1 \cdot K + C = \tfrac{1}{3} i + \tfrac{5}{2} j + K + C$$

$$\Rightarrow \; C = -\tfrac{1}{3} i - \tfrac{3}{2} j + K$$

$$r(t) = \left(\tfrac{1}{3} t^3\right) i + \left(\tfrac{5}{2} t^2\right) j + tK - \tfrac{1}{3} i - \tfrac{3}{2} j + K$$

$$= \frac{t^3 - 1}{3}\, i + \frac{5t^2 - 3}{2}\, j + (t+1) K$$