14.3 - 14.5 HW

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14.3 - Partial Derivs.

3. Quotient Rule : $\frac{d}{dy} \frac{y}{x+y} = \frac{(x+y)\frac{d}{dy}(y) - y\frac{d}{dy}(x+y)}{(x+y)^2} = \frac{(x+y)\cdot 1 - y \cdot 1}{(x+y)^2}$

$= x/(x+y)^2$

5. Calc. $f_z(2,3,1)$, where $f(x,y,z) = xyz$

$f_z(x,y,z) = d/dz\ (xyz) = xy$

$f_z(2,3,1) = 2 \cdot 3 = 6$

17. 1^{st} order partial deriv. : $z = x/y$

$d/dx\ (x/y) = 1/y \cdot d/dx\,(x) = 1/y \cdot 1 = 1/y$

$d/dy\ (x/y) = x \cdot d/dy\,(1/y) = x \cdot -1/y^2 = -x/y^2$

19. 1^{st} order partial deriv. : $z = \sqrt{9-x^2-y^2}$

$\frac{d}{dx}\left(\sqrt{9-x^2-y^2}\right) = \frac{1}{2\sqrt{9-x^2-y^2}} \frac{d}{dx}(9-x^2-y^2) = \frac{-2x}{2\sqrt{9-x^2-y^2}} = \frac{-x}{\sqrt{9-x^2-y^2}}$

$\frac{d}{dy}\left(\sqrt{9-x^2-y^2}\right) = \frac{1}{2\sqrt{9-x^2-y^2}} \frac{d}{dy}(9-x^2-y^2) = \frac{-2y}{2\sqrt{9-x^2-y^2}} = \frac{-y}{\sqrt{9-x^2-y^2}}$

21. 1^{st} order partial deriv. : $z = (\sin x)(\sin y)$

$d/dx\ (\sin x \cos y) = \cos y\ d/dx\ \sin x = (\cos x)(\cos y)$

$d/dy\ (\sin x \cos y) = \sin x\ d/dy\ \cos y = -(\sin x)(\sin y)$

27. 1^{st} order partial deriv. : $W = e^{r+s}$

$dW/dr = e^{r+s} \cdot d/dr\,(r+s) = e^{r+s} \cdot 1 = e^{r+s}$

$dw/ds = e^{r+s} \cdot d/ds\,(r+s) = e^{r+s} \cdot 1 = e^{r+s}$

31. 1^{st} order partial deriv. : $z = e^{-x^2-y^2}$

$dz/dx = e^{-x^2-y^2}\ d/dx\,(-x^2-y^2) = e^{-x^2-y^2} \cdot (-2x) = -2xe^{-x^2-y^2}$

$dz/dy = e^{-x^2-y^2}\ d/dy\,(-x^2-y^2) = e^{-x^2-y^2} \cdot (-2y) = -2ye^{-x^2-y^2}$

39. 1^{st} order partial deriv. : $Q = \frac{L}{M} e^{-Lt/M}$

$\frac{dQ}{dL} = \frac{d}{dL}\left(\frac{L}{M} e^{-Lt/M}\right) = \frac{L}{M} \cdot e^{-Lt/M} \cdot (-t/M) + e^{-Lt/M} \cdot \frac{1}{M}$

$= -\frac{Lt}{M^2} e^{-Lt/M} + \frac{e^{-Lt/M}}{M}$

$\frac{dQ}{dM} = \frac{d}{dM}\left(\frac{L}{M} e^{-Lt/M}\right) = \frac{L}{M} \cdot e^{-Lt/M} \cdot \frac{Lt}{M^2} + e^{-Lt/M} \cdot -\frac{L}{M^2}$

$= \frac{L^2 t}{M^3} e^{-Lt/M} - \frac{L}{M^2} e^{-Lt/M}$

$$\frac{dQ}{dt} = \frac{d}{dt}\left(\frac{L}{M} e^{-Lt/M}\right) = -\frac{L^2}{M^2} e^{-Lt/M}$$

11\. $I(T,H) = 45.33 + 0.6845T + 5.758H - 0.00365T^2 - 0.1565HT + 0.001HT^2$

(a) $I(95,50) = 45.33 + 0.6845(95) + 5.758(50) - 0.00365(95)^2 - 0.1565(50)(95) + 0.001(50)(95)^2 = 73.19125$

(b) $dI/dT = 0.6845 - 0.00730T - 0.1565H + 0.002HT$

$dI/dT(95,50) = 0.6845 - 0.00730(95) - 0.1565(50) + 0.002(50)(95) = 1.666$

14.4 - Differentiability, Tangent Planes, & Linear Approx.

For 3, 5, & 7, find eq. of tangent plane at given pt.

3\. $f(x,y) = x^2y + xy^3$, $(2,1)$

Eq.: $z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$

$f(x,y) = x^2y + xy^3 \Rightarrow f(2,1) = 4 + 2 = 6$

$f_x(x,y) = 2xy + y^3 \Rightarrow f_x(2,1) = 4 + 1 = 5$

$f_y(x,y) = x^2 + 3xy^2 \Rightarrow f_y(-1,2) = 1 - 12 = -11$

$z = 6 + 5(x-2) - 11(y-1) \Rightarrow z = 5x - 11y + 7$

5\. $f(x,y) = x^2 + y^{-2}$; $(4,1)$

Eq.: $f(4,1) + f_x(4,1)(x-4) + f_y(4,1)(y-1)$

$f(x,y) = x^2 + y^{-2} \Rightarrow f(4,1) = 17$

$f_x(x,y) = 2x \Rightarrow f_x(4,1) = 8$

$f_y(x,y) = -2y^{-3} \Rightarrow f_y(4,1) = -2$

$z = 17 + 8(x-4) - 2(y-1) = 8x - 2y - 13$

7\. $F(r,s) = r^2s^{-1/2} + s^{-3}$; $(2,1)$

$z = f(2,1) + f_r(2,1)(r-2) + f_s(2,1)(s-1)$

$f(r,s) = r^2s^{-1/2} + s^{-3} \Rightarrow f(2,1) = 5$

$f_r(r,s) = 2rs^{-1/2} \Rightarrow f_r(2,1) = 4$

$f_s(r,s) = -\frac{1}{2}r^2s^{-3/2} - 3s^{-4} \Rightarrow f_s(2,1) = -5$

$z = 5 + 4(r-2) - 5(s-1) = 4r - 5s + 2$

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13. Find linearization $L(x,y)$ of $f(x,y) = x^2y^3$ at $(a,b) = (2,1)$. Use it to est. $f(2.01, 1.02)$ & $f(1.97, 1.01)$ & compare w/ calc.

(a) $f(x,y) = x^2y^3 \Rightarrow f(2,1) = 4$

$f_x(x,y) = 2xy^3 \Rightarrow f_x(2,1) = 4$

$f_y(x,y) = 3x^2y^2 \Rightarrow f_y(2,1) = 12$

$f(x,y) \approx f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$

$f(x,y) \approx 4 + 4(x-2) + 12(y-1) = -16 + 4x + 12y$

(b) $h = x-2$ & $K = y-1$

$f(x,y) \approx f(2,1) + f_x(2,1)h + f_y(2,1)K = 4 + 4h + 12K$

$f(2.01, 1.02) \approx 4 + 4 \cdot 0.01 + 12 \cdot 0.02 = 4.28$

Calc.: $f(2.01, 1.02) = 2.01^2 \cdot 1.02^3 = 4.2874$

$h = 1.97 - 2 = -0.03$ & $K = 1.01 - 1 = 0.01$

$f(1.97, 1.01) \approx 4 + 4 \cdot (-0.03) + 12 \cdot 0.01 = 4$

Calc.: $f(1.97, 1.01) = 1.97^2 \cdot 1.01^3 = 3.998$

15. Let $f(x,y) = x^3y^{-4}$. Use eq. (5) to est. $\Delta f = f(2.03, 0.9) - f(2,1)$

$f(x,y) = x^3y^{-4} \Rightarrow f(2,1) = 8$

$f_x(x,y) = 3x^2y^{-4} \Rightarrow f_x(2,1) = 12$

$f_y(x,y) = -4x^3y^{-5} \Rightarrow f_y(2,1) = -32$

$\Delta x = 2.03 - 2 = 0.03$ & $\Delta y = 0.9 - 1 = 0.1$

$\Delta f = f(2.03, 0.9) - f(2,1) \approx f_x(2,1)\Delta x + f_y \Delta y$

$= 12 \cdot 0.03 + (-32) \cdot (-0.1) = 3.56 \Rightarrow \Delta f \approx 3.56$

17. Use linear approx. of $f(x,y) = e^{x^2+y}$ at $(0,0)$ to est. $f(0.01, -0.02)$. Compare w/ calc.

$f(h,K) \approx f(0,0) + f_x(0,0)h + f_y(0,0)K$

$f(x,y) = e^{x^2+y} \Rightarrow f(0,0) = e^0 = 1$

$f_x(x,y) = 2xe^{x^2+y} \Rightarrow f_x(0,0) = 2 \cdot 0 \cdot e^0 = 0$

$f_y(x,y) = e^{x^2+y} \Rightarrow f_y(0,0) = e^0 = 1$

$h = 0.01$ & $K = -0.02 \Rightarrow f(0.01, -0.02) \approx 1 + 0 \cdot 0.01 + 1 \cdot (-0.02) = 0.98$

Calc.: $f(0.01, -0.02) = e^{0.01^2 - 0.02} \approx 0.9803$

For 23, 25, 27, use linear approx. to est. val. Compare w/ calc.

23. $(2.01)^3(1.02)^2 \Rightarrow f(x,y) = x^3y^2$ at $(2,1)$

$f(2+h, 1+K) \approx f(2,1) + f_x(2,1)h + f_y(2,1)K$

$f(x,y) = x^3y^2 \Rightarrow f(2,1) = 8$
$f_x(x,y) = 3x^2y^2 \Rightarrow f_x(2,1) = 12$
$f_y(x,y) = 2x^3y \Rightarrow f_y(2,1) = 16$
$h = 0.01$ & $K = 0.02 \Rightarrow (2.01)^3(1.02)^2 \approx 8 + 12 \cdot 0.01 + 16 \cdot 0.02 = 8.44$
Calc.: $8.4487 \Rightarrow$ error $= 0.0087$ & % error $\approx (0.0087 \cdot 100)/8.4487 = 0.103\%$

25. $\sqrt{3.01^2 + 3.99^2} \Rightarrow f(x,y) = \sqrt{x^2+y^2}$ at $(3,4)$
$f(3+h, 4+K) \approx f(3,4) + f_x(3,4)h + f_y(3,4)K$
$f(x+y) = \sqrt{x^2+y^2} \Rightarrow f(3,4) = 5$
$f_x(x+y) = 2x/2\sqrt{x^2+y^2} = x/\sqrt{x^2+y^2} \Rightarrow f_x(3,4) = 3/5$
$f_y(x+y) = 2y/2\sqrt{x^2+y^2} = y/\sqrt{x^2+y^2} \Rightarrow f_y(3,4) = 4/5$
$h = 0.01$ & $K = -0.01 \Rightarrow \sqrt{3.01^2+3.99^2} \approx 5 + 3/5 \cdot 0.01 + 4/5 \cdot (-0.01) = 4.998$
Calc.: $\sqrt{3.01^2+3.99^2} \approx 4.99802 \Rightarrow$ error $= 0.00002$ & % error $\approx 0.0004002\%$

27. $\sqrt{(1.9)(2.02)(4.05)} \Rightarrow f(x,y,z) = \sqrt{xyz}$ at $(2,2,4)$
$f(2+h, 2+K, 4+l) \approx f(2,2,4) + f_x(2,2,4)h + f_y(2,2,4)K + f_z(2,2,4)l$
$f(x,y,z) = \sqrt{xyz}$ $\quad f(2,2,4) = 4$
$f_x(x,y,z) = yz/2\sqrt{xyz} = 1/2\sqrt{yz/x} \Rightarrow f_x(2,2,4) = 1$
$f_y(x,y,z) = xz/2\sqrt{xyz} = 1/2\sqrt{xz/y} \Rightarrow f_y(2,2,4) = 1$
$f_z(x,y,z) = xy/2\sqrt{xyz} = 1/2\sqrt{xy/z} \Rightarrow f_z(2,2,4) = 1/2$
$h = -0.1$ & $K = 0.02$ & $l = 0.05 \Rightarrow \sqrt{(1.9)(2.02)(4.05)} = 4 + 1 \cdot (-0.1) + 1 \cdot 0.02$
$+ 1/2\,(0.05) = 3.945$

Calc.: $\sqrt{(1.9)(2.02)(4.05)} \approx 3.9426$

14.5 - The Gradient & Directional Derivatives

7. $h(x,y,z) = xyz^{-3} \Rightarrow$ $\quad dh/dx = yz^{-3}$
$\nabla h = \langle dh/dx, dh/dy, dh/dz\rangle$ $\quad dh/dy = xz^{-3}$
$= \langle yz^{-3}, xz^{-3}, -3xyz^{-4}\rangle$ $\quad dh/dz = xy \cdot (-3z^{-4}) = -3xyz^{-4}$

11. $f(x,y) = x^2 - 3xy \Rightarrow d/dt\, f(r(t)) = \nabla f_{r(t)} \cdot r'(t)$
$\nabla f = \langle df/dx, df/dy\rangle = \langle 2x-3y, -3x\rangle$
$r'(t) = \langle -\sin t, \cos t\rangle$ $\quad t=0$; $r(t) = \langle \cos t, \sin t\rangle$, 2
$r(0) = \langle \cos 0, \sin 0\rangle = (1,0)$
$r'(0) = \langle -\sin 0, \cos 0\rangle = \langle 0,1\rangle$ $\quad d/dt\, f(r(t))|_{t=0} = \langle 2,3\rangle \cdot \langle 0,1\rangle = -3$
$\nabla f|_{r(0)} = \nabla f_{(1,0)} = \langle 2\cdot1 - 3\cdot0, -3\cdot1\rangle = \langle 2,-3\rangle$

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13. $f(x,y) = \sin(x,y)$, $r(t) = \langle e^{2t}, e^{3t} \rangle$, $t=0$

$d/dt\, f(r(t)) = \nabla f_{r(t)} \cdot r'(t) \Rightarrow \nabla f = \langle df/dx, df/dy \rangle = \langle y\cos(xy), x\cos(xy) \rangle$

$r(0) = (e^0, e^0) = (1,1)$ $\quad r'(t) = \langle 2e^{2t}, 3e^{3t} \rangle$

$r'(0) = \langle 2e^0, 3e^0 \rangle = \langle 2, 3 \rangle$

$\nabla f_{r(0)} = \nabla f_{(1,1)} = \langle \cos 1, \cos 1 \rangle$

$d/dt\, f(r(t)) \big|_{t=0} = \langle \cos 1, \cos 1 \rangle \cdot \langle 2, 3 \rangle = 5\cos 1$

19. $g(x,y,z) = xyz^{-1}$, $r(t) = \langle e^t, t, t^2 \rangle$, $t=1$

$d/dt\, g(r(t)) = \nabla g_{r(t)} \cdot r'(t) \Rightarrow \nabla g = \langle dg/dx, dg/dy, dg/dz \rangle = \langle yz^{-1}, xz^{-1}, -xyz^{-2} \rangle$

$r(1) = (e, 1, 1)$ $\quad r'(t) = \langle e^t, 1, 2t \rangle$

$r'(1) = \langle e, 1, 2 \rangle$

$\nabla g_{r(1)} = \nabla g_{(e,1,1)} = \langle 1, e, -e \rangle$

$d/dt\, g(r(t)) \big|_{t=1} = \langle 1, e, -e \rangle \cdot \langle e, 1, 2 \rangle = e + e - 2e = 0$

7. $f(x,y) = \ln(x^2+y^2)$, $v = 3i - 2j$, $P = (1,0)$

$u = v/\|v\| = 1/\sqrt{3^2+(-2)^2} \cdot (3i-2j) = (3i-2j)/\sqrt{13}$

$\nabla f = \langle df/dx, df/dy \rangle = \langle 2x/(x^2+y^2), 2y/(x^2+y^2) \rangle = 2\langle x,y \rangle/(x^2+y^2)$

$\nabla f_{(1,0)} = 2/(1^2+0^2) \cdot \langle 1, 0 \rangle = \langle 2, 0 \rangle = 2i$

$D_u f(1,0) = \nabla f_{(1,0)} \cdot u = 2i \cdot (3i-2j)/\sqrt{13} = 6/\sqrt{13}$

3. Find directional deriv. of $f(x,y) = x^2 + 4y^2$ at $P = (3,2)$ in the direction pointing to the origin

$v = \vec{PO} = \langle -3, -2 \rangle \Rightarrow u = v/\|v\| = \langle -3,-2 \rangle / \sqrt{3^2+2^2} = -\langle 3,2 \rangle/\sqrt{13}$

$\nabla f = \langle df/dx, df/dy \rangle = \langle 2x, 8y \rangle \Rightarrow \nabla f_{(3,2)} = \langle 6, 16 \rangle$

$D_u f(3,2) = \nabla f_{(3,2)} \cdot u = \langle 6, 16 \rangle \cdot -\langle 3, 2 \rangle/\sqrt{13} = -50/\sqrt{13}$

[illegible]. A bug located at (3, 9, 4) begins walking in a straight line towards (5, 7, 3). At what rate is the bug's temp. changing if the temp. is $T(x,y,z) = xe^{y-z}$

$v = \overline{PQ} = \langle 5-3, 7-9, 3-4 \rangle = \langle 2, -2, 1 \rangle$

$\nabla T = \langle T_x(x,y,z), T_y(x,y,z), T_z(x,y,z) \rangle = \langle e^{y-z}, xe^{y-z}, -xe^{y-z} \rangle$

$\nabla T_{(3,9,4)} = \langle e^{9-4}, 3e^{9-4}, -3e^{9-4} \rangle = \langle e^5, 3e^5, -3e^5 \rangle$

$D_u T(p) = \dfrac{1}{\|v\|} \nabla T_p \cdot v \Rightarrow D_u T(3,9,4) = \dfrac{1}{\|\langle 2,-2,-1 \rangle\|} \langle e^5, 3e^5, -3e^5 \rangle \cdot \langle 2, -2, -1 \rangle$

$= \dfrac{2e^5 - 6e^5 + 3e^5}{\sqrt{(2)^2+(-2)^2+(-1)^2}} = -\dfrac{e^5}{3} \approx -49.47\,°C$ per meter

37. Suppose that $\nabla f_P = \langle 2,-4,4\rangle$. Is f incr. or decr. at P in the direction $v=\langle 2,1,3\rangle$

$D_v f(P) = \nabla f_P \cdot v = \langle 2,-4,4\rangle \cdot \langle 2,1,3\rangle = 4-4+12 = 12 > 0$ (+ ; incr.)

39. Let $f(x,y,z) = \sin(xy+z)$ & $P = (0,-1,\pi)$. Calc. $D_u f(P)$, where u is a unit vector making an $\angle \theta = 30°$ w/ ∇f_P. $\Rightarrow D_u f(P) = \nabla f_P \cdot u$

$D_u f(P) = \|\nabla f_P\| \cdot \|u\| \cos 30° = \sqrt{3}\,\|\nabla f_P\|/2$

$\nabla f = \langle df/dx, df/dy, df/dz\rangle = \langle y\cos(xy+z), x\cos(xy+z), \cos(xy+z)\rangle = \cos(xy+z)\langle y,x,1\rangle$

$\nabla f_{(0,-1,\pi)} = \cos\pi\langle -1,0,1\rangle = -1\langle -1,0,1\rangle = \langle 1,0,-1\rangle$

$\|\nabla f_{(0,-1,\pi)}\| = \sqrt{1^2+0^2+(-1)^2} = \sqrt{2} \Rightarrow D_u f(P) = \sqrt{3}\cdot\sqrt{2}/2 = \sqrt{6}/2$

41. Find a vector normal to the surface $x^2+y^2-z^2=6$ at $P=(3,1,2)$

$f_x(x,y,z) = 2x$

$f_y(x,y,z) = 2y \Rightarrow \nabla f_P = \nabla f_{(3,1,2)} = \langle 6,2,-4\rangle$

$f_z(x,y,z) = -2z$

The vector $\langle 6,2,-4\rangle$ is normal to the surface $x^2+y^2-z^2=6$ at P.

43. Find 2 pts. on the ellipsoid $x^2/4 + y^2/9 + z^2 = 1$ where the tangent plane is normal to $v=\langle 1,1,-2\rangle$

constant

∇f_P & v are parallel $\Rightarrow \nabla f_P = Kv \Rightarrow \nabla f_P = \langle x/2, 2y/9, 2z\rangle = K\langle 1,1,-2\rangle$

$x = 2K,\ y = 9K/2,\ z = -K$

$$\frac{x^2}{4} + \frac{y^2}{9} + z^2 = \frac{(2K)^2}{4} + \frac{(9K/2)^2}{9} + (-K)^2 = 1$$

$K^2 + 9/4\,K^2 + K^2 = 1 \Rightarrow K = \pm 2/\sqrt{17}$

$$(x,y,z) = (2K, 9/2\,K, -K) = \pm\left(\frac{4}{\sqrt{17}}, \frac{9}{\sqrt{17}}, -\frac{2}{\sqrt{17}}\right)$$