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Exercise 14.6

Q1: $f(x,y,z)=x^2y^3+z^4$ $(x=s^2, y=st^2, z=s^2t)$

(a) calculate $\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz}$

$$\frac{df}{dx}=2xy^3 \qquad \frac{df}{dy}=3y^2x^2 \qquad \frac{df}{dz}=4z^3$$

(b) calculate $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$

$$\frac{dx}{ds}=2s, \quad \frac{dy}{ds}=t^2 \quad \frac{dz}{ds}=2st$$

(c) compute $\frac{df}{ds}$ using chain rule:

$$\frac{df}{ds}=\frac{df}{dx}\cdot\frac{dx}{ds}+\frac{df}{dy}\cdot\frac{dy}{ds}+\frac{df}{dz}\cdot\frac{dz}{ds}$$

$$=2xy^3\cdot 2s+3y^2x^2\cdot t^2+4z^3\cdot(2st)$$

$$=2s^2\cdot(st^2)^3\cdot 2s+3(st^2)^2\cdot s^4\cdot t^2+4(s^2t)^3\cdot(2st)$$

$$=4s^6\cdot t^6+3s^6t^6+8s^7t^4$$

$$=7s^6t^6+8s^7t^4$$

Q3. $\frac{df}{ds}, \frac{df}{dr}$; $f(x,y,z)=xy+z^2$, $x=s^2$, $y=2rs$, $z=r^2$

$$\frac{df}{ds}=\frac{df}{dx}\cdot\frac{dx}{ds}\cancel{=y\cdot 2s}+\frac{df}{dy}\cdot\frac{dy}{ds}+\frac{df}{dz}+\frac{dz}{ds}$$

$$=4rs^2+s^2\cdot 2r+2z\cdot 2r$$

$$=6rs^2+4r^3$$

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Q5. $\frac{dg}{du}, \frac{dg}{dv}$? $g(x,y) = \cos(x-y)$, $x = 3u-5v$, $y = -7u+15v$

$$\frac{dg}{du} = \frac{dg}{dx}\cdot\frac{dx}{du} + \frac{dg}{dy}\cdot\frac{dy}{du}$$

$$= (-\sin(x-y))\cdot 3 + (-1)(-\sin(x-y))\cdot(-7)$$

$$\cancel{= 3y\sin(x-y) + 7x\sin(x-y)}$$

$$\cancel{= \sin(x-y)(3y+7x)}$$

$$\cancel{= \sin}$$

$$= -\sin(x-y)(3+7)$$

$$= -10\sin(10u-20v)$$

$$\frac{dg}{dv} = \frac{dg}{dx}\cdot\frac{dx}{dv} + \frac{dg}{dy}\cdot\frac{dy}{dv}$$

$$= -\sin(x-y)\cdot(-5) + (-1)(-\sin(x-y))\cdot 15$$

$$= -\sin(x-y)\cdot(-20)$$

$$= 20\sin(10u-20v)$$

Q7: $\frac{df}{dy}$; $f(u,v) = e^{u+v}$, $u = x^2$, $v = xy$.

$$\frac{df}{dy} = \frac{df}{du}\cdot\frac{du}{dy} + \frac{df}{dv}\cdot\frac{dv}{dy}$$

$$= e^{u+v}\cdot 0 + e^{u+v}\cdot x$$

$$= xe^{x^2+xy}$$

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Q15. $\frac{dg}{du}$ at $(u,v)=(0,1)$, where $g(x,y)=x^2-y^2$
$x=e^u\cos v$, $y=e^u\sin v$.

$$\left.\frac{dg}{du}\right|_{(0,1)}=\frac{dg}{dx}\cdot\frac{dx}{du}+\frac{dg}{dy}\cdot\frac{dy}{du}=2x\cdot(e^u\cos v)+(-2y)\cdot(e^u\sin v)$$
$$=\cancel{2(e}\ 2e^0\cos 1\cdot(e^0\cos 1)$$
$$-2(e^0\sin 1)(e^0\sin 1)$$
$$=2\cos 2$$

~~Q17.~~ Q23. $x=s+t \quad y=s-t$

$$\left(\frac{df}{dx}\right)^2-\left(\frac{df}{dy}\right)^2=\frac{df}{ds}\cdot\frac{df}{dt}$$

$$\left(\frac{df}{dx}\right)^2-\left(\frac{df}{dy}\right)^2=\left(\frac{df}{dx}\cdot\frac{dx}{ds}+\frac{df}{dy}\cdot\frac{dy}{ds}\right)\cdot\left(\frac{df}{dx}\cdot\frac{dx}{dt}+\frac{df}{dy}\cdot\frac{dy}{dt}\right)$$

$$\left(\frac{df}{dx}\right)^2-\left(\frac{df}{dy}\right)^2=\cancel{\frac{df}{dx}}\left(\frac{df}{dx}\right)^2\cdot\frac{dx}{ds}\cdot\frac{dx}{dt}+\frac{df}{dx}\cdot\frac{dx}{ds}\cdot\frac{df}{dy}\cdot\frac{dy}{dt}$$
$$+\frac{df}{dy}\cdot\frac{dy}{ds}\cdot\frac{df}{dx}\cdot\frac{dx}{dt}+\left(\frac{df}{dy}\right)^2\cdot\frac{dy}{ds}\cdot\frac{dy}{dt}$$

$$\left(\frac{df}{dx}\right)^2-\left(\frac{df}{dy}\right)^2=\left(\frac{df}{dx}\right)^2\cdot 1+\frac{df}{dx}\cdot\frac{df}{dy}\cdot(-1)+\frac{df}{dy}\cdot\frac{df}{dx}+\left(\frac{df}{dy}\right)^2\cdot(-1)$$
$$=\left(\frac{df}{dx}\right)^2+(-1)\left(\frac{df}{dy}\right)^2$$

$$\therefore\left(\frac{df}{dx}\right)^2-\left(\frac{df}{dy}\right)^2=\left(\frac{df}{dx}\right)^2-\left(\frac{df}{dy}\right)^2$$

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Q27. $\frac{dz}{dx}$, $x^2y+y^2z+xz^2=10$

$\frac{dz}{dx}$: $z^2+2xy+\frac{dz}{dx}\cdot y^2+x2z\frac{dz}{dx}=0$

$$\frac{dz}{dx}(y^2+2z\cdot x)=-2xy-z^2$$

$$\frac{dz}{dx}=\frac{-2xy-z^2}{y^2+2zx}$$

Q29. $\frac{dz}{dy}$, $e^{xy}+\sin(xz)+y=0$

$$xe^{xy}+(x\frac{dz}{dy})\cdot\cos(xz)+1=0$$

$$\frac{dz}{dy}=\frac{-1-xe^{xy}}{x\cdot\cos(xz)}$$

Q31.

$\frac{dw}{dy}$. $\quad \frac{1}{w^2+x^2} + \frac{1}{w^2+y^2} = 1$

$$(w^2+x^2)^{-1} + (w^2+y^2)^{-1} = 1$$

$$\left(2w\cdot\frac{dw}{dy}\right)(-1)(w^2+x^2)^{-2} + \left(2w\cdot\frac{dw}{dy}+2y\right)(-1)\cdot(w^2+y^2)^{-2} = 0$$

$$-2w\frac{dw}{dy}(w^2+x^2)^{-2} + \left(-2w\frac{dw}{dy}\right)(w^2+y^2)^{-2} - 2y(w^2+y^2)^{-2} = 0$$

$$-2w\frac{dw}{dy}\left((w^2+x^2)^{-2} + (w^2+y^2)^{-2}\right) = 2y(w^2+y^2)^{-2}$$

$$\frac{dw}{dy} = \frac{2y(w^2+y^2)^{-2}}{\left[(w^2+x^2)^{-2}+(w^2+y^2)^{-2}\right](-2w)}$$

$$= \frac{y(w^2+y^2)^{-2}}{-\cancel{2}w\left[(w^2+x^2)^{-2}+(w^2+y^2)^{-2}\right]}$$

$$= \frac{y(w^2+y^2)^{2}}{-w\left[(w^2+x^2)^{2}+(w^2+y^2)^{+2}\right]} = \frac{4}{-(4+4)} = -\frac{1}{2}$$

$$\Rightarrow -\frac{1}{2}$$

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Exercise 14.7

Q1. $f(x,y)=x^2+y^4-4xy$

(a) $f_x(x,y)=2x-4y=0$ $x-2y=0$ $x=2y$ $2y=y^3$ $y=\sqrt{2}/-\sqrt{2}$

$f_y(x,y)=4y^3-4x=0$ $y^3=x$ $x=y^3$ $2=y^2$

$\therefore P=(0,0)$ $(2\sqrt{2},\sqrt{2})$ $(-2\sqrt{2},-\sqrt{2})$

(b). $f_{xx}=2$ $D=2(12y^2)-(-4)^2$

$f_{xy}=-4$ $=24y^2+(-16)$

$f_{yy}=12y^2$

① $y=0$ $D=-16$ neither max nor mini the saddle point is $(0,0)$

② $y=\sqrt{2}$ $D=32$ local minimum, the value is $f(2\sqrt{2},\sqrt{2})$ ~~$f(-2\sqrt{2}$~~ $=-4$

③ $y=-\sqrt{2}$ $D=32$

local minimum, the value is $f(-2\sqrt{2},-\sqrt{2})=-4$

Q3. $f(x,y)=8y^4+x^2+xy-3y^2-y^3$

$f_x=2x+y$ $2x+y=0$ $2x=-y$ $x=-\frac{y}{2}$

$f_y=32y^3+x-6y-3y^2$ $32y^3+(-\frac{y}{2})-6y-3y^2=0$

$y=0$ $x=0$; $y=-\frac{13}{32}$, $x=\frac{13}{64}$; $y=0.5$ $x=-\frac{1}{4}$

$f_{xx}=2$ $f_{xy}=1$ $f_{yy}=96y^2-6-6y$

$D=2(96y^2-6-6y)-(1)^2$

$(y=\frac{1}{2})=29$ $(y=0)=0-1=-1$ $(y=-\frac{13}{32})=\frac{377}{16}$

$\therefore$ at point $(0,0)$ it has saddle point.

at point $(\frac{13}{64},-\frac{13}{32})$ and $(-\frac{1}{4},\frac{1}{2})$ it has local minimum

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Q5. $f(x,y)=y^2x-yx^2+xy$

(a). $f_x=y^2-2xy+y$ | $y^2-2xy+y=0$ | $y(y-2x+1)=0$, $y=2xy-y^2$

$f_y=2yx-x^2+x$ | $2xy-x^2+x=0$ | ~~2x+2x~~ $x(2y-x+1)=0$.

$y^2+y-x^2+x=0$ when $x=\frac{1}{3}$, $y=-\frac{1}{3}$

(b) when $x=0, y=0$; when $x=1, y=0$; when $x=0, y=-1$

$\therefore$ critical point $(0,0)$ $(1,0)$ $(0,1)$ $(\frac{1}{3},-\frac{1}{3})$

(c). $f_{xx}=-2y$ $f_{xy}=2y-2x+1$ $f_{yy}=2x$

$D=(-2y)(2x)-(2y-2x+1)^2$

$$\left.\begin{aligned}(0,0)&=-1\\(1,0)&=-1\\(0,1)&=-9\end{aligned}\right\}\text{saddle point}$$

$(\frac{1}{3},-\frac{1}{3})=\frac{1}{3}>0$, $f_{xx}=\frac{2}{3}>0$

$\therefore$ local minimum at $(\frac{1}{3},-\frac{1}{3})$.

Q7. $f(x,y)=x^2+y^2-xy+x$

$f_x=2x-y+1$ | $2x-y+1=0$ | $4y-y+1=0$ $3y=-1$ $y=-\frac{1}{3}$

$f_y=2y-x$ | $2y=x$ | $x=-\frac{2}{3}$

$(-\frac{2}{3},-\frac{1}{3})$

$f_{xx}=2$

$f_{xy}=-1$

$f_{yy}=2$

$D=4-1=3>0$

$f_{xx}>0$ $\therefore$ at point $(-\frac{2}{3},-\frac{1}{3})$ is local minimum.

Q11. $f(x,y)=4x-3x^3-2xy^2$

$f_x=4-9x^2-2y^2$ | $4-9x^2-2y^2=0$ | when $x=0, y=\pm\sqrt{2}$

$f_y=-4xy$ | $-4xy=0$ | when $y=0$, $x=\pm\frac{2}{3}$

$(0,\sqrt{2})$ $(0,-\sqrt{2})$ $(\frac{2}{3},0)$ $(-\frac{2}{3},0)$ are critical points.

$f_{xx}=-18x$ $f_{xy}=-4y$ $f_{yy}=-4x$

$D=(-18x)(-4x)-(-4y)^2=72x^2-16y^2$

$$\left.\begin{aligned}(0,\sqrt{2})&=-32\\(0,\sqrt{2})&=-32\end{aligned}\right\}\text{saddle point.}$$

$(\frac{2}{3},0)=32$ $f_{xx}<0$ local maximum

$(-\frac{2}{3},0)=32$ $f_{xx}>0$ local minimum

Q13. $f(x,y) = x^4 + y^4 - 4xy$

$f_x = 4x^3 \cancel{\quad} - 4y$ $\quad 4x^3 - 4y = 0 \quad x^3 = y$ $\quad \therefore (x,y) \Rightarrow (0,0)$ $(1,1)$ $(-1,-1)$

$f_y = 4y^3 - 4x$ $\quad 4y^3 - 4x = 0 \quad x = y^3$

$f_{xx} = 12x^2 \quad f_{xy} = -4 \quad f_{yy} = 12y^2$

$D = (12x^2)(12y^2) - (-4)^2$

$(0,0) = -16 < 0$ saddle point.

$(1,1) = 128 > 0 \quad f_{xx} > 0$

$(-1,-1) = 128 > 0 \quad f_{xx} > 0$ $\}$ local minimum.

Q17. $f(x,y) = \sin(x+y) - \cos x$

$f_x = \cos(x+y) + \sin x$ $\quad \cos(x+y) + \sin x = 0$ $\quad \sin x = 0 \quad x = \pi, \pi \ldots$ $\quad \therefore x = j\pi$

$f_y = \cancel{s}\cos(x+y)$ $\quad \cos(x+y) = 0$

$f_{xx} = -\sin(x+y) + \cos x$ $\quad x+y=0 \quad y = k\pi + \frac{\pi}{2}$

$f_{xy} = -\sin(x+y)$

$f_{yy} = -\sin(x+y)$.

$\therefore$ j, k even, saddle point

j even k odd local minimum.

j odd, k even ~~local maximum~~ saddle point

j, k odd local maximum.

Q19. $f(x,y) = \ln x + 2\ln y - x - 4y$

$f_x = \frac{1}{x} - 1$ $\quad \frac{1}{x} - 1 = 0 \quad \frac{1}{x} = 1 \quad x = 1$

$f_y = \frac{2}{y} - 4$ $\quad \frac{2}{y} = 4 \quad y = \frac{1}{2}$ $\quad \therefore$ critical point is $(1, \frac{1}{2})$

$f_{xx} = -x^{-2} \quad f_{xy} = 0 \quad f_{yy} = -2y^{-2}$

$D = (-x^{-2})(-2y^{-2}) - 0$

$= 8$

$f_{xx} = -1 < 0$. $\quad \therefore$ at point $(1, \frac{1}{2})$ has local maximum.

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Q23. $f(x,y) = (x+3y)e^{y-x^2}$

$f_x = e^{y-x^2} + (x+3y)(-2x)e^{y-x^2}$

$f_y = 3e^{y-x^2} + (x+3y)e^{y-x^2}$

$-3e^{y-x^2} - 3(x+3y)(-2x)e^{y-x^2} + 3e^{y-x^2} + (x+3y)e^{y-x^2} = 0$

$-3(x+3y)(-2x)e^{y-x^2} + (x+3y)e^{y-x^2} = 0$

$(6x+1)(x+3y)e^{y-x^2} = 0$

$6x+1=0 \quad x=\frac{1}{-6}$; $x+3y=0$; $x=0,\ y=0.$

$y = \frac{7}{-18}$ $\quad\therefore$ critical point $= (0,0)\ (\frac{1}{-6}, \frac{7}{-18})$

$\cancel{f_{xx} = (-2x)e^{y-x^2} + (-2x)e^{y-x^2} + (x+3y)(4x^2)}$

$f_{xx} = (-2x)e^{y-x^2} + (-4x-6y)e^{y-x^2} + (-6xy-2x^2)(-2x)e^{y-x^2}$

≈ 2.143

$f_{xy} = e^{y-x^2} + (6x)e^{y-x^2} + (-6xy-2x^2)e^{y-x^2} \approx 0.630$

$f_{yy} = 3e^{y-x^2} + 3e^{y-x^2} + (x+3y)e^{y-x^2} \approx 1.134$

$D = 2.143 \times 1.134 - (0.63)^2 = 2.033 > 0.$

$f_{xx} > 0$

$\therefore (\frac{1}{-6}, \frac{7}{-18})$ local minimum.

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Q29. $f(x,y)=x+y,\ 0\le x\le 1,\ 0\le y\le 1.$

$(0,y)$ $0+0=0$ $0+1=1$

$(1,y)$ $1+0=1$ $1+1=2$

$(x,0)$ $0+0=0$ $0+1=1$

$(x,1)$ $1+0=1$ $1+1=2$

$\therefore$ Global maximum is 2

Global minimum is 0.

Q35. $f(x,y)=x+y-x^2-y^2-xy \quad 0\le x\le 2,\ 0\le y\le 2$

(a) $f_x=1-2x-y$ $\quad 1-2x-y=0$ $\quad 1-2y-x=0$

$f_y=1-2y-x$ $\quad y=1-2x$ $\quad 1-2(1-2x)-x=0$

$3x-1=0$

$x=\frac{1}{3}\ y=\frac{1}{3}$

the critical point is $(\frac{1}{3},\frac{1}{3})$, $f(\frac{1}{3},\frac{1}{3})=\frac{1}{3}$.

(b). $f(x,0)=x-x^2$

$x=\frac{1}{3},0,2.$

$\therefore f(x,0)_{x=\frac{1}{3}}=\frac{1}{3}-\frac{1}{9}=\frac{2}{9}$ $\cancel{\frac{1}{3}}$ ✓.

$f(x,0)_{x=0}=0$ $\quad f(x,0)_{\frac{1}{3}}=\frac{2}{9}$ bigger.

$f(x,0)_{x=2}=2-4=-2$

(c). $f(x,2)=x-x^2+y-y^2-xy$

$f(x,2)_{x=0}=2-4=-2$ ✓.

$f(x,2)_{x=\frac{1}{3}}=\frac{1}{3}-\frac{1}{9}+2-4-\frac{2}{3}=-2-\frac{1}{3}-\frac{1}{9}=-\frac{22}{9}$

$f(x,2)_{x=2}=2-4+2-4-4=-8$

$f(x,2)_{x=0}=-2$ bigger.

$f(0,y)=y-y^2 \quad y=0,\frac{1}{3},2$

$f(0,y)_0=0 \quad f(0,y)_{y=\frac{1}{3}}=\frac{1}{3}-\frac{1}{9}=\frac{2}{9}$ ✓

$f(0,y)_{y=2}=2-4=-2$ $\quad$ at $f=\frac{2}{9}$ is bigger.

$f(2,y)=x+y-x^2-y^2-xy$

~~$y=0$ $f(2,y)=2-4=-2$~~ ✓ $\quad$ ~~$\therefore f=-2$ bigger.~~

$y=\frac{1}{3}$ $f(2,y)=2+\frac{1}{3}-4-\frac{1}{9}-\frac{2}{3}=-\frac{22}{9}$

$y=2$ $f(2,y)=2+2-4-4-4=-8$

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(d). the maximum value of f is $\frac{1}{3}, \frac{2}{9}, -2, \frac{2}{9}, -2$

$\therefore$ the maximum value should be $\frac{1}{3}$.