14.6 Homework:

1) $f(x,y,z) = x^2y^3 + z^4$ and $x = s^2$, $y = st^2$, $z = s^2t$

a) $\boxed{\frac{\partial f}{\partial x} = 2xy^3,\ \frac{\partial f}{\partial y} = 3x^2y^2,\ \frac{\partial f}{\partial z} = 4z^3}$

b) $\boxed{\frac{\partial x}{\partial s} = 2s,\ \frac{\partial y}{\partial s} = t^2,\ \frac{\partial z}{\partial s} = 2st}$

c) $\boxed{\begin{aligned}\frac{\partial f}{\partial s} &= (2xy^3)(2s) + (3x^2y^2)(t^2) + (4z^3)(2st)\\ \frac{\partial f}{\partial s} &= 4sxy^3 + 3t^2x^2y^2 + 8stz^3\end{aligned}}$

3) $f(x,y,z) = xy + z^2$, $x = s^2$, $y = 2rs$, $z = r^2$

$\frac{\partial f}{\partial x} = y$, $\frac{\partial f}{\partial y} = x$, $\frac{\partial f}{\partial z} = 2z$

$\frac{\partial x}{\partial s} = 2s$, $\frac{\partial y}{\partial s} = 2r$, $\frac{\partial z}{\partial s} = 0$

$\frac{\partial x}{\partial r} = 0$, $\frac{\partial y}{\partial r} = 2s$, $\frac{\partial z}{\partial r} = 2r$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

$\frac{\partial f}{\partial s} = (y)(2s) + (x)(2r) + (2z)(0)$

$\frac{\partial f}{\partial s} = 2sy + 2rx$

$\frac{\partial f}{\partial r} = (y)(0) + (x)(2s) + (2z)(2r)$

$\frac{\partial f}{\partial r} = 2sx + 2rz$

$\boxed{\begin{aligned}\frac{\partial f}{\partial s} &= 6rs^2\\ \frac{\partial f}{\partial r} &= 2s^3 + 4r^3\end{aligned}}$

5) $g(x,y) = \cos(x-y)$, $x = 3u - 5v$, $y = -7u + 15v$

$\frac{\partial g}{\partial x} = -\sin(x-y)$, $\frac{\partial g}{\partial y} = \sin(x-y)$

$\frac{\partial x}{\partial u} = 3$, $\frac{\partial x}{\partial v} = -5$, $\frac{\partial y}{\partial u} = -7$, $\frac{\partial y}{\partial v} = 15$

$\frac{\partial g}{\partial u} = -3\sin(x-y) - 7\sin(x-y) = -10\sin(x-y)$

$\frac{\partial g}{\partial v} = 5\sin(x-y) + 15\sin(x-y) = 20\sin(x-y)$

$\frac{\partial g}{\partial u} = -10\sin(10u - 20v)$

$\frac{\partial g}{\partial v} = 20\sin(10u - 20v)$

7) $F(u,v) = e^{u+v}$, $u = x^2$, $v = xy$

$\frac{\partial F}{\partial u} = e^{u+v}$, $\frac{\partial F}{\partial v} = e^{u+v}$

$\frac{\partial u}{\partial y} = 0$, $\frac{\partial v}{\partial y} = x$

$\frac{\partial F}{\partial y} = (e^{u+v})(0) + (e^{u+v})(x)$

$\boxed{\frac{\partial F}{\partial y} = xe^{x^2+xy}}$

15) $(u,v) = (0,1)$ $g(x,y) = x^2 - y^2$, $x = e^u\cos v$, $y = e^u\sin v$

$\frac{\partial g}{\partial x} = 2x$, $\frac{\partial g}{\partial y} = -2y$

$\frac{\partial x}{\partial u} = e^u\cos v$, $\frac{\partial x}{\partial v} = e^u\cos v$, $\frac{\partial y}{\partial u} = e^u\sin v$, $\frac{\partial y}{\partial v} = e^u\sin v$

$\frac{\partial g}{\partial u} = (\cos(1))(2x) + (\sin(1))(-2y)$

$\frac{\partial g}{\partial u} = 2\cos^2(1) - 2\sin^2(1) = 2\cos(1+1)$

$\boxed{\frac{\partial g}{\partial u} = 2\cos(2)}$

17) d = distance between hitter and baseman

h = speed of hitter, b = speed of baseman

$h = h(t)$ $b = b(t)$

$d^2 = h^2 + b^2$

Solving for $\frac{\partial d}{\partial h}$

$\frac{d}{dh}(d^2 = h^2 + b^2)$

$2d\frac{\partial d}{\partial h} = 2h$

$\frac{\partial d}{\partial h} = \frac{h}{d}$

Solving for $\frac{\partial d}{\partial b}$

$\frac{d}{db}(d^2 = h^2 + b^2)$

$2d\frac{\partial d}{\partial b} = 2b$

$\frac{\partial d}{\partial b} = \frac{b}{d}$

Given $h = 8$, $b = 6$, $d = 10 \rightarrow (\sqrt{8^2+6^2})$

$\frac{\partial d}{\partial h} = \frac{4}{5}$, $\frac{\partial d}{\partial b} = \frac{3}{5}$, $\frac{\partial h}{\partial t} = 20$, $\frac{\partial b}{\partial t} = 18$

$\frac{\partial d}{\partial t} = \frac{\partial d}{\partial h}\frac{\partial h}{\partial t} + \frac{\partial d}{\partial b}\frac{\partial b}{\partial t}$

$\frac{\partial d}{\partial t} = (\frac{4}{5})(20) + (\frac{3}{5})(18)$

$\frac{\partial d}{\partial t} = 16 + \frac{54}{5}$

$\boxed{\frac{\partial d}{\partial t} = -26.8\ ft/s}$

(−) because d is getting smaller

23) $x = s + t$, $y = s - t$

$\frac{\partial x}{\partial s} = 1$, $\frac{\partial y}{\partial s} = 1$, $\frac{\partial x}{\partial t} = 1$, $\frac{\partial y}{\partial t} = -1$

$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$

$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}$

$\frac{\partial f}{\partial s}\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right)\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right)$

$\boxed{\frac{\partial f}{\partial s}\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2}$

27) $f(x,y) = x^2y + y^2z + xz^2 = 10$

Implicit diff rule: $\frac{\partial z}{\partial x} = -\frac{f_x}{f_z}$

$f_x = 2xy + z^2$

$f_z = y^2 + 2xz$

$\boxed{\frac{\partial z}{\partial x} = -\frac{2xy + z^2}{y^2 + 2xz}}$

29) $f(x,y) = e^{xy} + \sin(xz) + y = 0$

$f_y = xe^{xy} + 1$, $f_z = x\cos(xz) + 1$

$\boxed{\frac{\partial z}{\partial y} = -\frac{xe^{xy}+1}{x\cos(xz)+1}}$

31) $f(x,y,w) = \frac{1}{w^2 + x^2} + \frac{1}{w^2 + y^2} = 1$ @ $(x,y,z) = (1,1,1)$

$f_y = -2y(w^2+y^2)^{-2}$, $f_w = -2w(w^2+x^2)^{-2} - 2w(w^2+y^2)^{-2}$

$f_y = -\frac{1}{4}$, $f_w = -\frac{1}{2}$

$\frac{\partial w}{\partial y} = -(-\frac{1}{4})(-\frac{2}{1})$

$\boxed{\frac{\partial w}{\partial y} = -\frac{1}{2}}$

14.7 Homework

(1) $f(x,y) = x^2 + y^4 - 4xy$

$f_x = 2x - 4y$, $f_y = 4y^3 - 4x$

$2x - 4y = 0$, $4y^3 - 4x = 0$

Critical Points: $(0,0)$, $(-2\sqrt{2}, \sqrt{2})$, $(2\sqrt{2}, \sqrt{2})$

a) $f_x(a,b) = 2a - 4b = 0$

$2a = 4b = \boxed{a = 2b}$

$f_y(a,b) = 4b^3 - 4a = 0$

$= 4b^3 - 4(2b) = 0$

$= 4b^3 - 8b$

$= 4b(b^2 - 2) = 0$

$= b(b^2 - 2)$

$\boxed{b = 0, \pm\sqrt{2} \text{ , } a = 2b \text{ so } a = 0, \pm 2\sqrt{2}}$

b) $(0,0)$ is a saddle point
$(2\sqrt{2}, \sqrt{2})$ & $(-2\sqrt{2}, \sqrt{2})$ are local minima
Absolute min of $f = -4$

(3) $f(x,y) = 8y^4 + x^2 + xy - 3y^2 - y^3$

$f_x = 2x + y$, $f_y = 32y^3 + x - 6y - 3y^2$

$2x + y = 0$, $32y^3 + x - 6y - 3y^2 = 0$

Critical Points are: $(-\frac{1}{4}, \frac{1}{2})$, $(0,0)$, $(\frac{13}{64}, -\frac{13}{32})$
$(0,0)$ is a saddle point
$(-\frac{1}{4}, \frac{1}{2})$ is a local minima
$(\frac{13}{64}, -\frac{13}{32})$ is a local minima

(5) $f(x,y) = y^2x - yx^2 + xy$

$f_x = y^2 - 2yx + y$, $f_y = 2yx - x^2 + x$

$y^2 - 2yx + y = 0$, $2yx - x^2 + x = 0$

a)

Critical Points: $(1,0)$, $(0,-1)$, $(\frac{1}{3}, -\frac{1}{3})$, $(0,0)$

$y(y - 2x + 1) = 0$, $x(2y - x + 1) = 0$

$(1,0)$: $0(0 - 2 + 1) = 0$, $1(0 - 1 + 1) = 0$ ✓

$(0,-1)$: $-1(-1 - 0 + 1) = 0$, $0(-2 - 0 + 1) = 0$ ✓

$(\frac{1}{3}, -\frac{1}{3})$: $-\frac{1}{3}(-\frac{1}{3} - \frac{2}{3} + \frac{3}{3}) = 0$, $\frac{1}{3}(-\frac{2}{3} - \frac{1}{3} + \frac{3}{3}) = 0$ ✓

$(0,0)$: $0(0 - 0 + 1) = 0$, $0(0 - 0 + 1) = 0$ ✓

b) Critical Points: $(1,0)$, $(0,-1)$, $(\frac{1}{3}, -\frac{1}{3})$, $(0,0)$

(7) $f(x,y) = x^2 + y^2 - xy + x$

$f_x = 2x - y + 1$, $f_y = 2y - x$

Critical Points: $(-\frac{2}{3}, -\frac{1}{3})$

$f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = -1$

$D(-\frac{2}{3}, -\frac{1}{3}) = 2 \cdot 2 - 1 = 3 \quad 3 > 0$

$\boxed{(-\frac{2}{3}, -\frac{1}{3}) \text{ is a local minimum}}$

(11) $f(x,y) = 4x - 3x^3 - 2xy^2$

$f_x = 4 - 9x^2 - 2y^2$, $f_y = -4xy$

Critical Points: $(-\frac{2}{3}, 0)$, $(0, -\sqrt{2})$, $(0, \sqrt{2})$, $(\frac{2}{3}, 0)$

$f_{xx} = -18x$, $f_{yy} = -4x$, $f_{xy} = -4y$

$f_{xx}(-\frac{2}{3}, 0) = 12$, $f_{yy}(-\frac{2}{3}, 0) = \frac{8}{3}$, $f_{xy}(-\frac{2}{3}, 0) = 0$

$D(-\frac{2}{3}, 0) = 12 \cdot \frac{8}{3} - 0 = 32$ $\boxed{(-\frac{2}{3}, 0) \text{ is a local minimum}}$

$f_{xy}(0, -\sqrt{2}) = 0$, $f_{yy}(0, -\sqrt{2}) = 0$, $f_{xy}(0, -\sqrt{2}) = 4\sqrt{2}$

$D(0, -\sqrt{2}) = 0 \cdot 0 - 32$ $\boxed{(0, -\sqrt{2}) \text{ is a saddle point}}$

$f_{xx}(0, \sqrt{2}) = 0$, $f_{yy}(0, \sqrt{2}) = 0$, $f_{xy}(0, \sqrt{2}) = -4\sqrt{2}$

$D(0, \sqrt{2}) = 0 \cdot 0 - 32$ $\boxed{(0, \sqrt{2}) \text{ is a saddle point}}$

$f_{xx}(\frac{2}{3}, 0) = -12$, $f_{yy}(\frac{2}{3}, 0) = -\frac{8}{3}$, $f_{xy}(\frac{2}{3}, 0) = 0$

$D(\frac{2}{3}, 0) = -12 \cdot -\frac{8}{3} - 0 = 32$ $\boxed{(\frac{2}{3}, 0) \text{ is a local maximum}}$

(13) $f(x,y) = x^4 + y^4 - 4xy$

$f_x = 4x^3 - 4y$, $f_y = 4y^3 - 4x$

Critical Points: $(0,0)$, $(-1,-1)$, $(1,1)$

$f_{xx} = 12x^2$, $f_{yy} = 12y^2$, $f_{xy} = 4$

$D = f_{xx}f_{yy} - [f_{xy}]^2$

$D(1,1) = 128 > 0$ — $(1,1)$ is a local minimum

$D(-1,-1) = 128 > 0$ — $(-1,-1)$ is a local minimum

$D(0,0) = -16 > 0$ — $(0,0)$ is a saddle point

(17) $f(x,y) = \sin(x+y) - \cos x$

$f_x = \cos(x+y) + \sin x$, $f_y = \cos(x+y)$

$f_{xx} = -\sin(x+y) + \cos x$, $f_{yy} = -\sin(x+y)$, $f_{xy} = -\cos(x+y)$

Critical Points: $(a\pi, b\pi + \frac{\pi}{2})$

when $a, b \in E$	$D < 0$	they are saddle point
when $a, b \in O$	$D > 0$	they are local maxima
When $a \in E$, $b \in O$	$D > 0$	they are local minima
When $a \in O$, $b \in E$	$D > 0$	they are saddle points

c)

$f_{xx} = -2y$ $\quad f_{xx}(1,0) = 0$, $f_{xx}(0,-1) = -2$, $f_{xx}(\frac{1}{3}, -\frac{1}{3}) = -\frac{2}{3}$, $f_{xx}(0,0) = 0$

$f_{yy} = 2x$ $\quad f_{yy}(1,0) = 2$, $f_{yy}(0,-1) = 0$, $f_{yy}(\frac{1}{3}, -\frac{1}{3}) = \frac{2}{3}$, $f_{yy}(0,0) = 0$

$f_{xy} = 2y - 2x + 1$ $\quad f_{xy}(1,0) = -1$, $f_{xy}(0,-1) = -1$, $f_{xy}(\frac{1}{3}, -\frac{1}{3}) = -\frac{1}{3}$, $f_{xy}(0,0) = 1$

$D(1,0) = 0 \cdot 2 - 1 = -1 \quad -1 < 0$ $\boxed{(1,0) \text{ is a saddle point}}$

$D(\frac{1}{3}, -\frac{1}{3}) = \frac{2}{3} \cdot \frac{2}{3} - \frac{1}{9} = \frac{1}{3} \quad \frac{1}{3} > 0$ $\boxed{(\frac{1}{3}, -\frac{1}{3}) \text{ is a local minimum}}$

$D(0,-1) = 2 \cdot 0 - 1 = -1 \quad -1 < 0$ $\boxed{(0,-1) \text{ is a saddle point}}$

$D(0,0) = 0 \cdot 0 - 1 = -1 > 0$ $\boxed{(0,0) \text{ is a saddle point}}$

(19) $f(x,y) = \ln(x) + 2\ln(y) - x - 4y$

$f_x = \frac{1}{x} - 1$, $f_y = \frac{2}{y} - 4$

Critical Points: $(1, \frac{1}{2})$

$f_{xx} = -x^{-2}$, $f_{yy} = -2y^{-2}$, $f_{xy} = 0$

$f_{xx}(1, \frac{1}{2}) = -1$, $f_{yy}(1, \frac{1}{2}) = -8$, $f_{xy}(1, \frac{1}{2}) = 0$

$D(1, \frac{1}{2}) = -1 \cdot -8 - 0 = 8 > 0$

$\boxed{(1, \frac{1}{2}) \text{ is a local maximum}}$

(23) $f(x,y) = (x+3y)e^{y-x^2}$

$f_x = (1-2x^2-6xy)e^{y-x^2}$, $f_y = (3+x+3y)e^{y-x^2}$

Critical Point: $(-\frac{1}{6}, -\frac{17}{18})$

$f_{xx} = (2x^3+6x^2y-3x-3y)2e^{y-x^2}$,

$f_{yy} = (6+x+3y)e^{y-x^2}$,

$f_{xy} = (1-6xy-2x^2-6x)e^{y-x^2}$

$D(-\frac{1}{6}, -\frac{17}{18}) > 0$ & $f_{xx} > 0$

$\boxed{(-\frac{1}{6}, -\frac{17}{18}) \text{ is a local minimum}}$

(29) $f(x,y) = x+y$

$\boxed{\begin{array}{l}(0,0) \text{ is the global maximum} \\ (1,1) \text{ is the global minimum}\end{array}}$

(35) $f(x,y) = x+y-x^2-y^2-xy$, $0 \le x \le 2$, $0 \le y \le 2$

$f_x = 1-2x-y$, $f_y = 1-2y-x$

$f_{xx} = -2$, $f_{yy} = -2$, $f_{xy} = -1$

Critical Points: $\boxed{\text{The absolute maximum value is } \frac{1}{8}}$

$f(\frac{1}{3}, \frac{1}{3}) = \frac{1}{3}$

$f_x(x,0) = 1-2x \rightarrow (\frac{1}{2}, 0)$

$f(\frac{1}{2}, 0) = \frac{1}{4}$

$f_y(0,y) = 1-2y \rightarrow (0, \frac{1}{2})$

$f(0, \frac{1}{2}) = \frac{1}{4}$

$f_y(2,y) = -1-2y \rightarrow (2, -\frac{1}{2})$

$f(2, -\frac{1}{2}) = -\frac{7}{4}$

$f(x,2) = -1-2x \rightarrow (-\frac{1}{2}, 2)$

$f(-\frac{1}{2}, 2) = -\frac{7}{4}$