14.6

1. (a) $\frac{\partial f}{\partial x} = 2xy^3$

$\frac{\partial f}{\partial y} = 3x^2y^2$

$\frac{\partial f}{\partial z} = 4z^3$

(b) $\frac{\partial x}{\partial s} = 2s$

$\frac{\partial y}{\partial s} = t^2$

$\frac{\partial z}{\partial s} = 2st$

(c) $\frac{\partial f}{\partial s} = 4sxy^3 + 3t^2x^2y^2 + 8stz^3$

$= 4s^6t^6 + 3s^6t^6 + 8s^7t^4$

$= 7s^6t^6 + 8s^7t^4$

3. $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$

$\cancel{= (y+z^2)\cdot 2s + (x+z^2)\cdot 2r + 2z\cdot 0}$

$\cancel{= 2sy + 2sz^2 + 2rx + 2rz^2}$

$\cancel{= 4s^3r + 2sr^4 + 2rs^2 + 2r^5}$

$= 2s\cdot y + 2r\cdot x$

$= 4s^2r + 2s^2r$

$= 6s^2r.$

$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}.$

$= x\cdot 2s + 2z\cdot 2r$

$= 2s^3 + 4r^3$

5. $\frac{\partial g}{\partial u} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial u}$

$= -\sin(x-y)\cdot 3 + \sin(x-y)\cdot(-7)$

$= -10\sin(10u-20v)$

$\frac{\partial g}{\partial v} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial v}$

$= -\sin(x-y)\cdot(-5) + \sin(x-y)\cdot 15$

$= 20\sin(10u-20v)$

7. $\frac{\partial F}{\partial y} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y}$

$= e^{u+v}\cdot x$

$= x\cdot e^{x^2+xy}$

15. $\frac{\partial g}{\partial u} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial u}$

$= 2x\cdot\cos v e^u + (-2y)\cdot\sin v e^u$

$= 2(\cos v e^u)^2 - 2(\sin v e^u)^2$

when $(u, v) = 0, 1$.

$\frac{\partial g}{\partial u} = 2(\cos 1)^2 - 2(\sin 1)^2 = 2\cos 2$

23. $\cancel{\left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2}$

$\frac{\partial f}{\partial s}\frac{\partial f}{\partial t}$

$= \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}\right)\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\right)$

$= \left(\frac{\partial f}{\partial x}\right)^2\cdot\frac{\partial x}{\partial s}\frac{\partial x}{\partial t} + \left(\frac{\partial f}{\partial y}\right)^2\cdot\frac{\partial y}{\partial s}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial x}{\partial s}\frac{\partial y}{\partial t}$

$= \left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2 + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}$

$= \left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2.$

27. $\cancel{\frac{\partial f}{\partial x} = 2xy + z^2}$

$\cancel{\frac{\partial f}{\partial z} = y^2 + 2xz.}$

$\cancel{\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} \div \frac{\partial f}{\partial z}}$

$\cancel{= \frac{2xy+z^2}{y^2+2xz}}$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}$$

$$= 2xy + z^2 + (y^2+2xz)\frac{\partial z}{\partial x} = 0.$$

$$\frac{\partial z}{\partial x} = -\frac{2xy+z^2}{2xz+y^2}$$

29. $\cancel{\frac{\partial f}{\partial y} = \frac{\partial f}{\partial y}\frac{\partial x}{\partial y} + \frac{\partial f}{\partial y}}$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y}$$

$$= x\cdot e^{xy} + 1 + (x\cdot\cos(xz))\frac{\partial z}{\partial y} = 0.$$

$$\frac{\partial z}{\partial y} = -\frac{x\cdot e^{xy}+1}{x\cos(xz)}$$

31. $$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y}$$

$$= -(w^2+y^2)^{-2}\cdot 2y + \left(-(w^2+x^2)^{-2}\cdot 2w - (w^2+y^2)^{-2}\cdot 2w\right)\frac{\partial w}{\partial y}$$

$\cancel{= \frac{2y}{(w^2+y^2)^2} - \frac{2w}{(w^2+y^2)^2(w^2+x^2)^2}}$

$$= -\frac{2y}{(w^2+y^2)^2} - \left\{2w\cdot\left[(w^2+x^2)^{-2} + (w^2+y^2)^{-2}\right]\right\}\frac{\partial w}{\partial y} = 0$$

$$\frac{\partial w}{\partial y} = \frac{-y(w^2+y^2)^{-2}}{w(w^2+x^2)^{-2} + w\cdot(w^2+y^2)^{-2}}$$

when $(x, y, w) = (1, 1, 1)$

$$\frac{\partial w}{\partial y} = \frac{-\frac{1}{4}}{\frac{1}{2}} = -\frac{1}{2}.$$

14.7

1(a) $f_x(x,y) = 2x - 4y = 0$

$2a = 4b$

$a = 2b.$

$f_y(x,y) = 4y^3 - 4x = 0.$

$\cancel{x = y^3}$ $\begin{cases} a = b^3 \\ a = 2b \end{cases}$

$b^3 = 2b$

① $b = 0$, $(a,b) = (0,0)$

② $b \neq 0$, $b^2 = 2$

$b = \pm\sqrt{2}.$

$(a,b) = (2\sqrt{2}, \sqrt{2})$ or $(-2\sqrt{2}, -\sqrt{2})$

(b) From the figure,

$(0,0)$ is saddle point.

when $(2\sqrt{2}, \sqrt{2})$ $f(x,y) = -4$

when $(-2\sqrt{2}, -\sqrt{2})$ $f(x,y) = -4.$

These two are local. min.

the absolute min. is -4

3. $f_x(x,y) = 2x + y$

$f_y(x,y) = 32y^3 + x - 6y - 3y^2$

when they are 0.

$y = -2x$

$x = 6y + 3y^2 - 32y^3.$

the critical points are

$(0,0), (-\frac{1}{4}, \frac{1}{2}), (\frac{13}{64}, -\frac{13}{32})$

From the figure, $(0,0)$ is saddle

$(-\frac{1}{4}, \frac{1}{2}), (\frac{13}{64}, -\frac{13}{32})$ are local min

5(a) $f_x(x,y) = y^2 - 2xy + y = y(y-2x+1) = 0.$

$f_y(x,y) = 2xy - x^2 + x = x(2y-x+1) = 0.$

(b) ① $y=0.$

$x\cdot(1-x)=0.$

$(0,0), (1,0)$

② $x=0.$

$y\cdot(y+1)=0$

$(0,0), (0,-1)$

③ $x\neq 0, y\neq 0.$

$\begin{cases} y-2x+1=0 \\ 2y-x+1=0. \end{cases}$

$y=2x-1=\frac{x-1}{2}$

$4x-2=x-1.$

$x=\frac{1}{3}; y=-\frac{1}{3}.$

∴ the critical points are $(0,0), (1,0), (0,-1), (\frac{1}{3},-\frac{1}{3}).$

(c). $f_{xx}(x,y) = -2y$

$f_{yy}(x,y) = 2x.$

$f_{xy}(x,y) = 2y-2x+1$

$D(0,0) = -4xy-(2y-2x+1)^2 = -1<0.$

∴ $(0,0)$ is saddle point.

$D(1,0)=-1<0.$

∴ $(1,0)$ is saddle point.

$D(0,-1)=-1<0.$

∴ $(0,-1)$ is saddle point.

$D(\frac{1}{3},-\frac{1}{3})=\frac{4}{9}-\frac{1}{9}=\frac{1}{3}>0.$

∵ $f_{xx}(\frac{1}{3},-\frac{1}{3})=\frac{2}{3}>0.$

∴ $(\frac{1}{3},-\frac{1}{3})$ is local min.

7. $f_x(x,y)=2x-y+1.$

$f_y(x,y)=2y-x$

If both of them are 0.

$x=2y$

$3y+1=0$

$(-\frac{2}{3},-\frac{1}{3})$ is the point.

$f_{xx}(x,y)=2$

$f_{yy}(x,y)=2$

$f_{xy}(x,y)=-1.$

$D(-\frac{2}{3},-\frac{1}{3})=4-1=3>0.$

$f_{xx}(-\frac{2}{3},-\frac{1}{3})=2>0.$

∴ $(-\frac{2}{3},-\frac{1}{3})$ is local min.

11. $f_x(x,y)=4-9x^2-2y^2.$

$f_y(x,y)=-4xy$

① $x=0.\ y^2=2,\ y=\pm\sqrt{2}.$

② $y=0,\ x^2=\frac{4}{9},\ x=\pm\frac{2}{3}.$

$(0,\sqrt{2}), (0,-\sqrt{2}), (\frac{2}{3},0), (-\frac{2}{3},0).$ are critical points.

$f_{xx}(x,y)=-18x.$

$f_{yy}(x,y)=\cancel{-4y}-4x.$

$f_{xy}(x,y)=-4y.$

$D(0,\sqrt{2})=-18<0.$

$D(0,-\sqrt{2})=-18<0.$

$D(\frac{2}{3},0)=32>0.$

$f_{xx}(\frac{2}{3},0)=-12<0.$

$D(-\frac{2}{3},0)=32>0.$

$f_{xx}(-\frac{2}{3},0)=12>0.$

∴ $(0,\sqrt{2})(0,-\sqrt{2})$ are saddle points

∴ $(\frac{2}{3},0)$ is local max. $(-\frac{2}{3},0)$ local min.

13. $f_x(x,y) = 4x^3 - \cancel{4y^2}\, 4y$

$f_y(x,y) = 4y^3 - 4x$

If both of them are 0.

$\begin{cases} y = x^3 \\ x = y^3 \end{cases}$

$x^9 = x$.

① $x=0$. $(0,0)$.

② $x \neq 0$, $(1,1)$, $(-1,-1)$

$\therefore (0,0), (1,1), (-1,-1)$ are critical points

$f_{xx}(x,y) = 12x^2$

$f_{yy}(x,y) = 12y^2$.

$f_{xy}(x,y) = -4$.

$D(0,0) = -16 < 0$.

$D(1,1) = 144 - 16 = 128 > 0$.

$f_{xx}(1,1) = 12 > 0$.

$D(-1,-1) = 144 - 16 = 128 > 0$.

$f_{xx}(-1,-1) = 12 > 0$.

$\therefore (0,0)$ is saddle point.

$(1,1)$ and $(-1,-1)$ are local min.

19. $f_x(x,y) = \frac{1}{x} - 1$.

$f_y(x,y) = \frac{2}{y} - 4$.

$(1, \frac{1}{2})$ is the only critical point.

$f_{xx}(x,y) = -x^{-2}$

$f_{yy}(x,y) = -2y^{-2}$.

$f_{xy}(x,y) = 0$.

$D(1, \frac{1}{2}) = (-1) \cdot (-8) = 8 > 0$.

$f_{xx}(x,y) = -1 < 0$.

$\therefore (1, \frac{1}{2})$ is local max.

21. $f_x(x,y) = 1 - \frac{1}{x+y} \cancel{= \frac{y}{x+y}}$

$f_y(x,y) = -2y - \cancel{\frac{1}{x+y}}\, \frac{1}{x+y}$

If both of them ~~is~~ are 0.

$\begin{cases} x+y = 1 \\ y = -\frac{1}{2} \end{cases}$

$\therefore$ ~~x = 2~~ $(\frac{3}{2}, \frac{1}{2})$ is the only critical point.

$f_{xx}(x,y) = (x+y)^{-2}$

$f_{yy}(x,y) = -2 + (x+y)^{-2}$.

$f_{xy}(x,y) = (x+y)^{-2}$.

$D(\frac{3}{2}, \frac{1}{2}) = \frac{1}{4} \cdot (-\frac{7}{4}) - \frac{1}{16} = -\frac{1}{2} < 0$.

$\therefore (\frac{3}{2}, \frac{1}{2})$ is saddle point.

23. $f_x(x,y) = e^{y-x^2} \cdot (1 - 2x^2 - 6xy)$

$f_y(x,y) = e^{y-x^2} \cdot (3 + x + 3y)$

If both of them are 0.

$\begin{cases} 1 - 2x^2 - 6xy = 0 \\ 3 + \cancel{3x}\, x + 3y = 0 \end{cases}$.

$2x(x+3y) = 1$.

$x + 3y = -3$

$x = -\frac{1}{6}$, $y = -\frac{17}{18}$

$\therefore (-\frac{1}{6}, -\frac{17}{18})$ is the only critical point

$f_{xx}(x,y) = e^{y-x^2} \cdot (-6x + 4x^3 + 12x^2 \ldots - 6y \ldots$

$f_{yy}(x,y) = e^{y-x^2}(6 + x + 3y)$

$f_{xy}(x,y) = e^{y-x^2}(1 - 2x^2 - 6xy - 6x)$

$D(-\frac{1}{6}, -\frac{17}{18}) = \cancel{\frac{134}{9} e^{-\frac{35}{18}}}\ \frac{53}{18} e^{-\frac{35}{18}} - e^{-\frac{35}{18}}$

$= \frac{35}{18} e^{-\frac{35}{18}} > 0$.

$f_{xx}(-\frac{1}{6}, -\frac{17}{18}) = \frac{53}{54} e^{-\frac{35}{36}} > 0$.

$\therefore (-\frac{1}{6}, -\frac{17}{18})$ is local min.

29. $f_x(x,y)=1$.

$f_y(x,y)=1$.

There are no critical point.

① On left side.

$x=0$

$f(0,y)=y$

~~f(y)=1~~

$f(0,0)=0$ is abs. min on left side

$f(0,1)=1$ is abs. max on left side

② on right side

$x=1$

$f(1,y)=y+1$.

$f(1,0)=1$ is abs. min on ~~left~~ right side

$f(1,1)=2$ is abs. max on right side

③ on down side

$y=0$

$f(x,0)=x$.

$f(0,0)=0$ is abs. min on down side

$f(1,0)=1$ is abs. max on down side

~~f(1,1)~~

④ on up side

$y=1$

$f(x,1)=x+1$.

$f(0,1)=1$ is abs. min on up side

$f(1,1)=2$ is abs. max on up side.

$\therefore$ 0 is abs. min

2 is abs. max.

35. $f_x(x,y)=1-2x-y$

$f_y(x,y)=1-2y-x$.

when they are 0.

$$\begin{cases} y+2x=1 \\ x+2y=1. \end{cases}$$

$x=y=\frac{1}{3}$.

$\therefore (\frac{1}{3},\frac{1}{3})$ is critical point. $f(\frac{1}{3},\frac{1}{3})=\frac{1}{3}$.

① on left side

$x=0$

$f(0,y)=y-y^2$.

$f(0,0)=0$

$f(0,\frac{1}{3})=\frac{1}{9}$ is abs. max. on left side

$f(0,2)=-2$ is ~~tae~~ abs. min on left side

② on right side

$x=2$

$f(2,y)=-y-y^2-2$ $f'(2,y)=-2y-1$

$f(2,0)=-2$ is abs. max on right side

~~$f(2,\frac{1}{3})=$~~ $f(2,2)=-8$ is abs. min on right side

③ on down side

$y=0$

$f(x,0)=x-x^2$

$f(0,0)=0$

$f(\frac{1}{3},0)=\frac{1}{9}$ is abs. max on down side

$f(2,0)=-2$ is abs. min. on down side

④ On up side

$y=2$

$f(x,2)=-x-x^2-2$ $f'(x,2)=-2x-1$.

$f(0,2)=-2$ is abs. max on up side

$f(2,2)=-8$ is abs. min on up side

$\therefore$ ~~-8 is abs. min.~~

~~$\frac{1}{9}$ is abs. max.~~

$\frac{1}{3}$ is abs. max.