14.6

1. $f(x,y,z) = x^2y^3 + z^4$, $x = s^2$, $y = st^2$, $z = s^2t$

a. $\frac{df}{dx} = 2xy^3$ $\quad \frac{df}{dy} = 3y^2x^2$ $\quad \frac{df}{dz} = 4z^3$

b. $\frac{dx}{ds} = 2s$ $\quad \frac{dy}{ds} = t^2$ $\quad \frac{dz}{ds} = 2st$

c. $\frac{df}{ds} = (2xy^3)(2s) + (3y^2x^2)(t^2) + (4z^3)(2st)$

$\frac{df}{ds} = 2(s^2)(st^2)^3\, 2s + 3(st^2)^2(s^2)^2(t^2) + 4(s^2t)^3(2st)$

$\frac{df}{ds} = 4s^6t^6 + 3s^6t^6 + 6s^7t^4$

$\frac{df}{ds} = 7s^6t^6 + 8s^7t^4$

3. $f(x,y,z) = xy + z^2$, $x = s^2$, $y = 2rs$, $z = r^2$

$\frac{df}{dx} = y$ $\quad \frac{df}{dy} = x$ $\quad \frac{df}{dz} = 2z$

$\frac{dx}{ds} = 2s$ $\quad \frac{dy}{ds} = 2r$ $\quad \frac{dz}{ds} = 0$

$\frac{dx}{dr} = 0$ $\quad \frac{dy}{dr} = 2s$ $\quad \frac{dz}{dr} = 2r$

$\frac{df}{ds} = 2sy + 2rx$ $\quad \frac{df}{dr} = 2sx + 4rz$

$\frac{df}{ds} = 2s(2rs) + 2r(s^2)$ $\quad \frac{df}{dr} = 2s(s^2) + 4r(r^2)$

$\frac{df}{ds} = 4rs^2 + 2rs^2 = 6rs^2$ $\quad \frac{df}{dr} = 2s^3 + 4r^3$

5. $g(x,y) = \cos(x-y)$, $x = 3u - 5v$, $y = -7u + 15v$

$\frac{dg}{dx} = -\sin(x-y) \cdot 1$ $\quad \frac{dg}{dy} = -\sin(x-y) \cdot -1 = \sin(x-y)$

$\frac{dx}{du} = 3$ $\quad \frac{dy}{du} = -7$ $\quad \frac{dx}{dv} = -5$ $\quad \frac{dy}{dv} = 15$

$\frac{dg}{du} = -3\sin(x-y) - 7\sin(x-y) = -3\sin(10u - 20v) - 7\sin(10u - 20v)$

$\frac{dg}{dv} = 5\sin(x-y) + 15\sin(x-y) = 5\sin(10u-20v) + 15\sin(10u-20v)$

7. $F(u,v) = e^{u+v}$, $u = x^2$, $v = xy$

$\frac{dF}{du} = e^{u+v}$ $\frac{dF}{dv} = e^{u+v}$ $\frac{du}{dx} = 2x$ $\frac{dv}{dx} = y$

$\frac{du}{dy} = 0$ $\frac{dv}{dy} = x$

$\frac{dF}{dy} = e^{u+v}(0) + e^{u+v}(x) = xe^{u+v}$

$\frac{dF}{dy} = xe^{x^2+xy}$

15. $\frac{dg}{du}$

$\frac{dg}{dx} = 2x$ $\frac{dg}{dy} = -2y$ $\frac{dx}{du} = e^u\cos v + e^u \sin v \cdot 0 = e^u \cos v$

$\frac{dy}{du} = e^u \sin v + e^u \cos v \cdot 0 = e^u \sin v$

$\frac{dg}{du} = 2x(e^u\cos v) + -2y(e^u \sin v)$

$\frac{dg}{du} = 2(e^u\cos v)(e^u\cos v) - 2(e^u\sin v)(e^u \sin v)$

$\frac{dg}{du} = 2(e^u\cos v)^2 - 2(e^u \sin v)^2 = 2\cos 2$

23. $x = s+t$ $y = s-t$

$\frac{dx}{ds} = 1$ $\frac{dy}{ds} = 1$ $\frac{dx}{dt} = 1$ $\frac{dy}{dt} = -1$

$\left(\frac{df}{dx}\right)^2 - \left(\frac{df}{dy}\right)^2 = \left(\frac{df}{dx} \cdot \frac{dx}{ds} + \frac{df}{dy} \cdot \frac{dy}{ds}\right) \left(\frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} \cdot \frac{dy}{dt}\right)$

$\left(\frac{df}{dx}\right)^2 - \left(\frac{df}{dy}\right)^2 = \left(\frac{df}{dx} + \frac{df}{dy}\right) + \left(\frac{df}{dx} - \frac{df}{dy}\right)$

$\left(\frac{df}{dx}\right)^2 - \left(\frac{df}{dy}\right)^2 = \left(\frac{df}{dx}\right)^2 - \left(\frac{df}{dy}\right)^2$

27. $\frac{dz}{dx}$, $x^2y + y^2z + xz^2 = 10$

$\cancel{x^2 \frac{dy}{dx}} + 2xy + 2y\cancel{\frac{dy}{dx}} \cdot z + y^2 \frac{dz}{dx} + x2z\frac{dz}{dx} + z^2 = 0$

$\frac{dz}{dx}(y^2 + x2z) = -2xy - z^2$

$\frac{dz}{dx} = \frac{-2xy - z^2}{y^2 + x2z}$

29. $\frac{dz}{dy}$, $e^{xy} + \sin(xz) + y = 0$

$$e^{xy} + \cos((xz)\cdot\left(x\frac{dz}{dy} + 0\right) + 1 = 0$$

$$\cos(xz)\cdot x\frac{dz}{dy} = -1 - e^{xy}$$

$$\frac{dz}{dy} = \frac{-1 - e^{xy}}{x\cos(xz)}$$

31. $\frac{dw}{dy}$, $\frac{1}{w^2+x^2} + \frac{1}{w^2+y^2} = 1$ $(1,1,1)$

$$\left(\frac{-1}{(w^2+x^2)^2}\cdot 2w\cdot\frac{dw}{dy} + 0\right) + \left(\frac{-1}{(w^2+y^2)^2}\cdot 2w\frac{dw}{dy} + 2y\right) = 0$$

$$2w\frac{dw}{dy}\left(\frac{-1}{(w^2+x^2)^2}\right) + 2w\frac{dw}{dy}\left(\frac{-1}{(w^2+y^2)^2}\right) + 2y\left(\frac{-1}{(w^2+y^2)^2}\right) = 0$$

$$2(1)\frac{dw}{dy}\left(\frac{-1}{4}\right) + 2(1)\frac{dw}{dy}\left(\frac{-1}{4}\right) + 2(1)\left(\frac{-1}{4}\right) = 0$$

$$\frac{dw}{dy} = \frac{\frac{2}{4}}{\frac{-2}{4}} = -1$$

14.7

1. (a,b) critical point of $f(x,y) = x^2 + y^4 - 4xy$

a. $f_x = 2x - 4y$ | $f_y = 4y^3 - 4x$ | $a = 2(0)$ $a = 2(\sqrt{2})$ $a = 2(-\sqrt{2})$

$0 = 2x - 4y$ | $0 = 4y^3 - 4x$ | $a = 0$ $a = 2\sqrt{2}$ $a = -2\sqrt{2}$

$4y = 2x$ | $0 = 4y^3 - 4(2y)$ | $(0,0)$ $(2\sqrt{2}, \sqrt{2})$ $(-2\sqrt{2}, -\sqrt{2})$

$2y = x \rightarrow a = 2b$ | $0 = 4y^3 - 8y$

$0 = 4y$ $\quad 0 = y^2 - 2$

$y = 0$ $\quad y = \sqrt{2}, -\sqrt{2}$

b. saddle point at $(0,0)$ local min $(2\sqrt{2}, \sqrt{2})$, $(-2\sqrt{2}, -\sqrt{2})$,
absolute minimum $f = -4$

3. $f(x,y) = 8y^4 + x^2 + xy - 3y^2 - y^3$

$f_x = 2x + y$ | $f_y = 32y^3 + x - 6y - 3y^2$

$f_{xx} = 2$ | $f_{yy} = 32y^3 - 3y^2 - 6y + x$

$f_{xy} = 1$

$0 = 2x + y$ | $0 = 32y^3 - 3y^2 - \cancel{6}\frac{12}{2}y - \frac{y}{2}$

$-y = 2x$ | $0 = y(32y^2 - 3y - \frac{13}{2})$

$-\frac{1}{2}y = x$ | $0 = y$ $\quad 0 = 32y^2 - 3y - \frac{13}{2}$

-208; -16, 13; -3

$-\frac{1}{2}(\frac{13}{32}) = x$ $\quad -\frac{1}{2}(\frac{1}{2}) = x$

$x = \frac{-13}{64}$ $\quad x = \frac{-1}{4}$ | $(32y^2 - 16y)(13y - \frac{13}{2}) = 0$

$(\frac{-13}{64}, \frac{-13}{32})$ $(\frac{-1}{4}, \frac{1}{2})$ $(0,0)$ | $16y(2y-1) + \frac{13}{2}(2y-1) = 0$

minima $\quad$ saddle | $(16y + \frac{13}{2})(2y-1) = 0$

$y = \frac{13}{32}$ $\quad y = \frac{1}{2}$

5. $f(x,y) = y^2x - yx^2 + xy$

a. $f_x = y^2 - 2xy + y$ | $f_y = 2yx - x^2 + x$

$0 = y^2 - 2xy + y$ | $0 = 2yx - x^2 + x$

$0 = y(y - 2x + 1)$ | $0 = x(2y - x + 1)$

$(\frac{1}{3}, -\frac{1}{3})$ $(0,0)$

$(\ldots)$ $(1,0)$

b. $0 = y$ $\quad 0 = y - 2x + 1$ | $0 = x$ $\quad 0 = 2y - x + 1$

$y = 2x - 1$ $\quad y = 2(0) - 1$ | $0 = 4x - 2 - x + 1$ [illegible] $+1$

$y = 2(\frac{1}{3}) - 1$ $\quad y = -1$

$y = -\frac{1}{3}$

1\. c. $f_{xx} = -2y$ $\quad f_{yy} = 2x$ $\quad f_{xy} = 2y - 2x + 1$

$D = -2y \cdot 2x - (2y - 2x + 1)^2$

$D(0,0) = -2(0) \cdot 2(0) - (2(0) - 2(0) + 1)^2 = -1$

$D(\frac{1}{3}, \frac{-1}{3}) = -2(\frac{-1}{3}) \cdot 2(\frac{1}{3}) - (2(\frac{-1}{3}) - 2(\frac{1}{3}) + 1)^2 = \frac{4}{9} - (\frac{1}{9}) = \frac{3}{9}$

$D(0,-1) = -2(-1) \cdot 2(0) - (2(-1) - 2(0) + 1)^2 = -(-2+1)^2 = -1$

$D(1,0) = -2(0) \cdot 2(1) - (2(0) - 2(1) + 1)^2 = -1$

$(0,0)$, $(0,-1)$, $(1,0)$ are saddle points and $(\frac{1}{3}, \frac{-1}{3})$ is local minimum

7\. $f(x,y) = x^2 + y^2 - xy + x$

$f_x = 2x - y + 1$
$0 = 2(2y) - y + 1$
$0 = 3y + 1$
$-1 = 3y$
$\frac{-1}{3} = y$

$f_y = 2y - x$
$0 = 2y - x$
$x = 2y$
$x = 2(\frac{-1}{3})$
$x = \frac{-2}{3}$

$f_{xx} = 2$
$f_{yy} = 2$
$f_{xy} = -1$

$D(\frac{-2}{3}, \frac{-1}{3}) = 2 \cdot 2 - (-1)^2 = 3$

$(\frac{-2}{3}, \frac{-1}{3})$ is a local minima

11\. $f(x,y) = 4x - 3x^3 - 2xy^2$

$f_x = 4 - 9x^2 - 2y^2$
$0 = 4 - 9x^2 - 2y^2$
$2y^2 = 4 - 9x^2$
$y = \sqrt{2 - \frac{9}{2}x^2}$
$y = \sqrt{2 - 0}$
$y = \sqrt{2}$ $\quad (0, \sqrt{2})$

saddle

$f_y = -4xy$
$0 = -4xy$
$0 = xy$

$0 = \sqrt{2 - \frac{9}{2}x^2}$ $\quad x = 0$ $\quad y = 0$
$2 = \frac{9}{2}x^2$
$\sqrt{\frac{4}{9}} = \sqrt{x^2}$
$\frac{2}{3} = x$ $\quad (\frac{2}{3}, 0)$ $\quad (\frac{-2}{3}, 0)$

max $\quad$ min

13. $f(x,y) = x^4 + y^4 - 4xy$

$f_x = 4x^3 - 4y$
$0 = 4x^3 - 4y$
$0 = x^3 - y$
$y = x^3$
$y = 0^3$ $y = 1^3$ $y = -1^3$
$y = 0$ $y = 1$ $y = -1$

$f_y = 4y^3 - 4x$
$0 = y^3 - x$
$0 = (x^3)^3 - x$
$0 = x^9 - x$
$0 = x$ $0 = x^8 - 1$
$\sqrt[8]{1} = x^8$
$x = 1, -1$

$(0,0)$ $(1,1)$
$(-1,-1)$

$f_{xx} = 12x^2$
$f_{yy} = 12y^2$
$f_{xy} = -4$

$D(0,0) = 12(0^2) \cdot 12(0^2) + (-4^2) = -16$
$D(1,1) = 12(1^2) \cdot 12(1^2) - (-4^2) = 128$
$D(-1,-1) = 12(-1^2) \cdot 12(-1^2) - (-4^2) = 128$
$(0,0)$ saddle point, $(1,1)$ and $(-1,-1)$ local min.

19. $f(x,y) = \ln x + 2\ln y - x + 4y$

$f_x = \frac{1}{x} - 1$
$0 = \frac{1}{x} - 1$
$1 = \frac{1}{x}$
$x = 1$

$f_y = \frac{2}{y} - 4$
$0 = \frac{2}{y} + 4$
$4 = \frac{2}{y}$
$4y = 2$
$y = \frac{1}{2}$

$f_{xx} = \frac{-1}{x^2}$
$f_{yy} = \frac{-2}{y^2}$
$f_{xy} = 0$

$D = \left(\frac{-1}{x^2}\right) \cdot \left(\frac{-2}{y^2}\right) - 0^2$
$D(1, \frac{1}{2}) = \frac{-1}{1} \cdot \frac{-2}{\frac{1}{4}} = -1 \cdot \frac{-8}{1} = 8$

$(1, \frac{1}{2})$ local max

17. $f(x,y) = \sin(x+y) - \cos x$

$f_x = \cos(x+y) \cdot 1 + \sin x$
$0 = \cos(x+y) + \sin x$
$\sin x = 0$
$0, \pi, \pi j$

$f_y = \cos(x+y) \cdot 1 + \sin y$
$0 = \cos(x+y) + \sin y$
$0 = \cos(\pi j + y)$
$(k\pi + \frac{\pi}{2})$

23. $f(x,y) = (x+3y)e^{y-x^2}$

$f_x = (x+3y)(-2e^{y-x^2})x + (1)e^{y-x^2}$

$f_y = (x+3y)(e^{y-x^2}) + 3(e^{y-x^2})$

$0 = (x+3y)(-2xe^{y-x^2}) + e^{y-x^2}$
$0 = (-2x^2-6xy+1)(e^{y-x^2})$
$-2x^2-6xy+1=0$ $\quad e^{y-x^2}=0$
$-2x(x-3y)=-1$ $\quad y-x^2=\ln 0$
$-2x=-1$ $\quad x-3y=-1$ $\quad$ UND.
$x=\frac{1}{2}$ $\quad x=3y-1$

$0=(x+3y)(e^{y-x^2})+3(e^{y-x^2})$
$0=(x+3y+3)(e^{y-x^2})$
$0=x+3y+3$ $\quad e^{y-x^2}=0$
$0=(3y-1)+3y+3$ $\quad$ UND.
$0=6y+2$
$-2=6y$
$\frac{-1}{3}=y$

$x=3(\frac{-1}{3})-1$
$x=0$
$(0, \frac{-1}{3})$

$0=\frac{1}{2}+3y+\frac{\cancel{3}\,6}{2}$
$\frac{-7}{2}=3y$
$\frac{-7}{6}=y$
$(\frac{1}{2}, \frac{-7}{6})$

29. $f(x,y)=x+y$ $\quad 0 \le x \le 1$ $\quad 0 \le y \le 1$ $\quad$ no calc
when f will be largest is when $x+y$ is largest @ $(1,1)$
$f(1,1)=2$ extreme max

when f is smallest $x+y$ is smallest @ $(0,0)$ $f(0,0)=0$ min

35. $f(x,y) = x + y - x^2 - y^2 - xy$ $0 \le x \le 2$ $0 \le y \le 2$

a. Down side $y = 0$ $0 \le x \le 2$

$f(x,0) = x - x^2$ $f(0) = 0$ $f(2) = 2 - 4 = -2$

$0 = x - x^2$

$x = 0$ $1 - x = 0$ Abs. Max $= 1$ Abs Min $= -2$

$1 = x$

Up side $y = 2$ $0 \le x \le 2$

$f(x,2) = x + 2 - x^2 - 4 - 2x$

$0 = -x^2 - x - 2$ $f(0) = -2$ $f(2) = -4 - 2 - 2 = -8$

$0 = x^2 + x + 2$

$-2 = x(x+1)$ Abs. Max $= -2$ Abs Min $= -8$

$x = -2$ $x = -3$

Right side $x = 2$ $0 \le y \le 2$

$f(2,y) = 2 + y - 4 - y^2 - 2y$

$0 = -2 - y - y^2$ $f(0) = -2$ $f(2) = -2 - 2 - 4 = -8$

$0 = y^2 + y + 2$

$-2 = y(y+1)$ Abs. Max $= -2$ Abs. Min $= -8$

$y = -2$ $y = -3$

Left side $x = 0$ $0 \le y \le 2$

$f(0,y) = y - y^2$

$0 = y - y^2$ $f(0) = 0$ $f(2) = -2$

$0 = y$ $0 = 1 - y$

$y = 1$ Abs. Max $= 1$ Abs. Min $= -2$

Abs. Max $= 1$