14.6 Homework

① $f(x,y,z) = x^2y^3 + z^4$ and $x = s^2$, $y = st^2$, and $z = s^2t$

$\to$ (a) $\frac{\partial f}{\partial x} = 2xy^3$, $\frac{\partial f}{\partial y} = 3x^2y^2$, $\frac{\partial f}{\partial z} = 4z^3$

$\to$ (b) $\frac{\partial x}{\partial s} = 2s$, $\frac{\partial y}{\partial s} = t^2$, $\frac{\partial z}{\partial s} = 2st$

$\to$ (c) $\frac{\partial f}{\partial s} = (2xy^3)\cdot(2s) + (3x^2y^2)\cdot(t^2) + (4z^3)\cdot(2st) = 4xy^3s + 3x^2y^2t^2 + 8z^3st$

$\to$ $\boxed{\frac{\partial f}{\partial s} = 7s^6t^6 + 8s^7t^4}$

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③ $\frac{\partial f}{\partial s}$, $\frac{\partial f}{\partial r}$; $f(x,y,z) = xy + z^2$, $x = s^2$, $y = 2rs$, $z = r^2$

$\to$ $\frac{\partial f}{\partial x} = y$, $\frac{\partial f}{\partial y} = x$, $\frac{\partial f}{\partial z} = 2z$

$\to$ $\frac{\partial x}{\partial s} = 2s$, $\frac{\partial y}{\partial s} = 2r$, $\frac{\partial z}{\partial s} = 0$

$\to$ $\frac{\partial x}{\partial r} = 0$, $\frac{\partial y}{\partial r} = 2s$, $\frac{\partial z}{\partial r} = 2r$

$\to$ $\frac{\partial f}{\partial s} = (y)\cdot(2s) + (x)\cdot(2r) + (2z)\cdot(0) = 2ys + 2xr = 4rs^2 + 2s^2r = 6rs^2$

$\to$ $\boxed{\frac{\partial f}{\partial s} = 6rs^2}$

$\to$ $\frac{\partial f}{\partial r} = (y)\cdot(0) + (x)(2s) + (2z)(2r) = 2xs + 4zr = 2s^3 + 4r^3$

$\to$ $\boxed{\frac{\partial f}{\partial r} = 2s^3 + 4r^3}$

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⑤ $\frac{\partial g}{\partial u}$, $\frac{\partial g}{\partial v}$; $g(x,y) = \cos(x-y)$, $x = 3u - 5v$, $y = -7u + 15v$

$\to$ $\frac{\partial g}{\partial x} = -\sin(x-y)$, $\frac{\partial g}{\partial y} = \sin(x-y)$

$\to$ $\frac{\partial x}{\partial u} = 3$, $\frac{\partial y}{\partial u} = -7$

$\to$ $\frac{\partial x}{\partial v} = -5$, $\frac{\partial y}{\partial v} = 15$

$\to$ $\frac{\partial g}{\partial u} = (-\sin(x-y))\cdot(3) + (\sin(x-y)\cdot(-7)) = -3\sin(x-y) - 7\sin(x-y) = -10\sin(x-y)$

$\to$ $-10\sin(3u - 5v + 7u - 15v) = -10\sin(10u - 20v)$

$\to$ $\boxed{\frac{\partial g}{\partial u} = -10\sin(10u - 20v)}$

$\to$ $\frac{\partial g}{\partial v} = (-\sin(x-y))(-5) + (\sin(x-y))(15) = 20\sin(x-y) = 20\sin(10u - 20v)$

$\to$ $\boxed{\frac{\partial g}{\partial v} = 20\sin(10u - 20v)}$

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(7) $\frac{\partial F}{\partial y}$; $F(u,v) = e^{u+v}$, $u = x^2$, $v = xy$

$\rightarrow \frac{\partial F}{\partial u} = e^{u+v}$, $\frac{\partial F}{\partial v} = e^{u+v}$

$\rightarrow \frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 0$

$\rightarrow \frac{\partial v}{\partial x} = y$, $\frac{\partial v}{\partial y} = x$

$\rightarrow \boxed{\frac{\partial F}{\partial y} = xe^{x^2+xy}}$

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(15) $\frac{dg}{du}$ at $(u,v) = (0,1)$ where $g(x,y) = x^2 - y^2$, $x = e^u \cos v$, $y = e^u \sin v$

$\rightarrow \frac{dg}{dx} = 2x$, $\frac{dg}{dy} = -2y$

$\rightarrow \frac{dx}{du} = e^u \cos v$, $\frac{dy}{du} = e^u \sin v$

$\rightarrow \frac{dx}{dv} = -e^u \sin(v)$, $\frac{dy}{dv} = e^u \cos(v)$

$\rightarrow \frac{dg}{du} = (2x)(e^u \cos v) + (-2y)(e^u \sin v) = 2\cos(1)(\cos(1)) + (-2\sin(1))(\sin(1)) = 2\cos^2(1) - 2\sin^2(1) = 2\cos(2)$

$\rightarrow \boxed{\frac{dg}{du} = 2\cos(2)}$

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(17) $\rightarrow \frac{\partial d}{\partial h} = \frac{h}{d}$

$\rightarrow \frac{\partial d}{\partial b} = \frac{b}{d}$

$\rightarrow \frac{\partial d}{\partial h} = \frac{4}{5}$, $\frac{\partial d}{\partial b} = \frac{3}{5}$, $\frac{\partial h}{\partial t} = 20$, $\frac{\partial b}{\partial t} = 18$

$\rightarrow \frac{\partial d}{\partial t} = \left(\frac{4}{5}\right)(20) + \left(\frac{3}{5}\right)(18)$

$\rightarrow \boxed{\frac{\partial d}{\partial t} = -26.8 \text{ ft/s}}$

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(23) $x = s + t$, $y = s - t$

$\rightarrow \frac{dx}{ds} = 1$, $\frac{dy}{ds} = 1$, $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = -1$

$\rightarrow \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} = \frac{\partial f}{\partial x} + \frac{\partial t}{\partial y}$

$\rightarrow \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$

$\rightarrow \boxed{\frac{\partial f}{\partial s} \cdot \frac{\partial f}{\partial t} = \left(\frac{\partial t}{\partial x}\right)^2 - \left(\frac{\partial t}{\partial y}\right)^2}$

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27 $f(x,y) = x^2y + y^2z + xz^2 = 10$

$\rightarrow f_x = 2xy + z^2$

$\rightarrow f_z = y^2 + 2xz$

$\rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{-2xy + z^2}{y^2 + 2xz}}$

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29 $f(x,y) = e^{xy} + \sin(xz) + y = 0$

$\rightarrow f_y = xe^{xy} + 1$

$\rightarrow f_z = x\cos(xz) + 1$

$\rightarrow \boxed{\frac{\partial z}{\partial y} = -\frac{xe^{xy} + 1}{x\cos(xz) + 1}}$

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31 $f(x,y,w) = \frac{1}{w^2+x^2} + \frac{1}{w^2+y^2} = 1$ at $(x,y,z) = (1,1,1)$

$\rightarrow f_y = -2y(w^2+y^2)^{-2}, \quad f_w = -2w(w^2+x^2)^{-2} - 2w(w^2+y^2)^{-2}$

$\rightarrow f_y = -\frac{1}{4}, \quad f_w = -\frac{1}{2}$

$\rightarrow \boxed{\frac{dw}{dy} = \frac{-1}{2}}$

14.7 Homework

① $f(x,y) = x^2 + y^4 - 4xy$

$\to$ (a) $f_x(x,y) = 0 = 2x - 4y$

$\to$ $2x - 4y = 0 \Rightarrow 2x = 4y \Rightarrow 2a = 4b \Rightarrow \boxed{a = 2b}$

$\to$ $f_y(x,y) = 4y^3 - 4x \Rightarrow 2x - 4y = 0;\ 4y^3 - 4x = 0 \Rightarrow \boxed{(0,0),\ (2\sqrt{2}, \sqrt{2}),\ (-2\sqrt{2}, \sqrt{2})}$

$\to$ (b) $(0,0)$ is a saddle point; $(2\sqrt{2}, \sqrt{2})$ and $(-2\sqrt{2}, -\sqrt{2})$ are local minima; $y = -4$ is the abs. min

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③ $f(x,y) = 8y^4 + x^2 + xy - 3y^2 - y^3$

$\to$ $f_x(x,y) = 2x + y$

$\to$ $f_y(x,y) = 32y^3 + x - 6y - 3y^2 \Rightarrow (0,0),\ \left(\frac{13}{64}, \frac{-13}{32}\right),\ \left(\frac{-1}{4}, \frac{1}{2}\right)$

$\to$ $(0,0)$ is a saddle point and $\left(\frac{13}{64}, \frac{-13}{32}\right)$, $\left(\frac{-1}{4}, \frac{1}{2}\right)$ are local minima

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⑤ $f(x,y) = y^2x - yx^2 + xy$

$\to$ $f_x(x,y) = y^2 - 2yx + y$

$\to$ $f_y(x,y) = 2yx - x^2 + x$

$\to$ Critical Points: $\left(\frac{1}{3}, \frac{-1}{3}\right),\ (0,-1),\ (0,0),\ (1,0)$

$\to$ (a) $y(y - 2x + 1) = 0 \Rightarrow -\frac{1}{3}\left(-\frac{1}{3} - 2\cdot\frac{1}{3} + 1\right) = 0 \Rightarrow \frac{1}{9} + \frac{2}{9} - \frac{3}{9} = 0 \Rightarrow 0 = 0$ ✓

$\to$ $y(y - 2x + 1) = 0 \Rightarrow -1(-1 - \cancel{2\cdot 0} + 1) = 0 \Rightarrow 1 - 1 = 0 \Rightarrow 0 = 0$ ✓

$\to$ $x(2y - x + 1) = 0 \Rightarrow \frac{1}{3}\left(-\frac{2}{3} - \frac{1}{3} + 1\right) = 0 \Rightarrow -\frac{2}{9} - \frac{1}{9} + \frac{3}{9} = 0 \Rightarrow 0 = 0$ ✓

$\to$ $x(2y - x + 1) = 0 \Rightarrow 0(2\cdot(-1) - 0 + 1) = 0 \Rightarrow 0 = 0$ ✓

$\to$ (b) Critical Points: $\left(\frac{1}{3}, \frac{-1}{3}\right),\ (0,-1),\ (0,0),\ (1,0)$

$\to$ $f_{xx} = -2y$

$\to$ $f_{xy} = 2y - 2x + 1$

$\to$ $f_{yy} = 2x$

$\to$ $f_{xx}\left(\frac{1}{3}, \frac{-1}{3}\right) = \frac{2}{3}$, $f_{xx}(0,-1) = 2$, $f_{xx}(0,0) = 0$, $f_{xx}(1,0) = 0$

$\rightarrow f_{xy}\left(\frac{1}{3}, -\frac{1}{3}\right) = -\frac{1}{3}$, $f_{xy}(0,-1) = -1$, $f_{xy}(0,0) = 1$, $f_{xy}(1,0) = -1$

$\rightarrow f_{yy}\left(\frac{1}{3}, -\frac{1}{3}\right) = \frac{2}{3}$, $f_{yy}(0,-1) = 0$, $f_{yy}(0,0) = 0$, $f_{yy}(1,0) = 2$

$\rightarrow D\left(\frac{1}{3}, -\frac{1}{3}\right) = \frac{2}{3} \cdot \frac{2}{3} - \frac{1}{9} = \frac{3}{9} = \frac{1}{3} > 0$ and $f_{xx}\left(\frac{1}{3}, -\frac{1}{3}\right) > 0$

$\rightarrow$ <u>$\left(\frac{1}{3}, -\frac{1}{3}\right)$ is a local minimum</u>

$\rightarrow D(0,-1) = 2 \cdot 0 - 1 = -1 < 0$

$\rightarrow$ <u>$(0,-1)$ is a saddle point</u>

$\rightarrow D(0,0) = 0 \cdot 0 - 1 = -1 < 0$

$\rightarrow$ <u>$(0,0)$ is a saddle point</u>

$\rightarrow D(1,0) = 0 \cdot 2 - 1 = -1 < 0$

$\rightarrow$ <u>$(1,0)$ is a saddle point</u>

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⑦ $f(x,y) = x^2 + y^2 - xy + x$

$\rightarrow f_x = 2x - y + 1$

$\rightarrow f_y = 2y - x$

$\rightarrow$ Critical Points: $\left(-\frac{2}{3}, -\frac{1}{3}\right)$

$\rightarrow f_{xx} = 2$

$\rightarrow f_{xy} = -1$

$\rightarrow f_{yy} = 2$

$\rightarrow D = 4 - 1 = 3 > 0$ and $f_{xx} > 0$

$\rightarrow$ <u>$\left(-\frac{2}{3}, -\frac{1}{3}\right)$ is a local minimum</u>

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⑪ $f(x,y) = 4x - 3x^3 - 2xy^2$

$\rightarrow f_x = 4 - 9x^2 - 2y^2$

$\rightarrow f_y = -4xy$

$\rightarrow$ Critical Points: $\left(-\frac{2}{3}, 0\right)$, $(0, -\sqrt{2})$, $(0, \sqrt{2})$, $\left(\frac{2}{3}, 0\right)$

$\to f_{xx} = -18x$; $f_{xx}\left(-\frac{2}{3}, 0\right) = 12$; $f_{xx}(0, -\sqrt{2}) = 0$; $f_{xx}(0, \sqrt{2}) = 0$, $f_{xx}\left(\frac{2}{3}, 0\right) = -12$

$\to f_{xy} = -4y$; $f_{xy}\left(-\frac{2}{3}, 0\right) = 0$; $f_{xy}(0, -\sqrt{2}) = 4\sqrt{2}$; $f_{xy}(0, \sqrt{2}) = -4\sqrt{2}$, $f_{xy}\left(\frac{2}{3}, 0\right) = 0$

$\to f_{yy} = -4x$; $f_{yy}\left(-\frac{2}{3}, 0\right) = \frac{8}{3}$; $f_{yy}(0, -\sqrt{2}) = 0$; $f_{yy}(0, \sqrt{2}) = 0$; $f_{yy}\left(\frac{2}{3}, 0\right) = -\frac{8}{3}$

$\to D\left(-\frac{2}{3}, 0\right) = 12 \cdot \frac{8}{3} - 0 = 32 > 0$ and $f_{xx}\left(-\frac{2}{3}, 0\right) > 0$

$\to$ $\left(-\frac{2}{3}, 0\right)$ is a local minimum

$\to D(0, -\sqrt{2}) = 0 \cdot 0 - 32 = -32 < 0$

$\to$ $(0, -\sqrt{2})$ is a saddle point

$\to D(0, \sqrt{2}) = 0 \cdot 0 - 32 = -32 < 0$

$\to$ $(0, \sqrt{2})$ is a saddle point

$\to D\left(\frac{2}{3}, 0\right) = -12 \cdot \frac{8}{3} - 0 = 32 < 0$ and $f_{xx}\left(\frac{2}{3}, 0\right) < 0$

$\to$ $\left(\frac{2}{3}, 0\right)$ is a local maximum

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(13) $f(x,y) = x^4 + y^4 - 4xy$

$\to f_x = 4x^3 - 4y$

$\to f_y = 4y^3 - 4x$

$\to$ Critical Points: $(-1,-1)$, $(0,0)$, $(1,1)$

$\to f_{xx} = 12x^2$; $f_{xx}(-1,-1) = 12$; $f_{xx}(0,0) = 0$; $f_{xx}(1,1) = 12$

$\to f_{xy} = -4$

$\to f_{yy} = 12y^2$; $f_{yy}(-1,-1) = 12$; $f_{yy}(0,0) = 0$; $f_{yy}(1,1) = 12$

$\to D(-1,-1) = 144 - 16 = 128 > 0$ and $f_{xx}(-1,-1) > 0$

$\to$ $(-1,-1)$ is a local minimum

$\to D(0,0) = 0 \cdot 0 - 16 = -16 < 0$

$\to$ $(0,0)$ is a saddle point

$\to D(1,1) = 144 - 16 = 128 > 0$ and $f_{xx}(1,1) > 0$

$\to$ $(1,1)$ is a local minimum

(17) $f(x,y) = \sin(x+y) - \cos x$

$\rightarrow f_x = \sin(x+y) + \sin x$

$\rightarrow f_y = \sin(x+y)$

$\rightarrow$ Critical Points: $\left(j\pi, K\pi + \frac{\pi}{2}\right)$

$\rightarrow$ When j and K are even: saddle points

$\rightarrow$ When j and K are odd: local maxima

$\rightarrow$ When j is even and K is odd: local minima

$\rightarrow$ When j is odd and K is even: saddle points

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(19) $f(x,y) = \ln(x) + 2\ln(y) - x - 4y$

$\rightarrow f_x = \frac{1}{x} - 1$

$\rightarrow f_y = \frac{2}{y} - 4$

$\rightarrow$ Critical Points: $\left(1, \frac{1}{2}\right)$

$\rightarrow f_{xx} = -\frac{1}{x^2}$; $f_{xx}\left(1, \frac{1}{2}\right) = -1$

$\rightarrow f_{xy} = 0$

$\rightarrow f_{yy} = -\frac{2}{y^2}$; $f_{yy}\left(1, \frac{1}{2}\right) = -8$

$\rightarrow D = 8 - 0 = 8 > 0$ and $f_{xx}\left(1, \frac{1}{2}\right) < 0$

$\rightarrow \left(1, \frac{1}{2}\right)$ is a local maximum

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(23) $f(x,y) = (x+3y)e^{y-x^2}$

$\rightarrow f_x = (1 - 2x^2 - 6xy)e^{y-x^2}$

$\rightarrow f_y = (3 + x + 3y)e^{y-x^2}$

$\rightarrow$ Critical Points: $\left(\frac{-1}{6}, -\frac{17}{18}\right)$

$\rightarrow f_{xx} = (2x^3 + 6x^2y - 3x - 3y)\,2e^{y-x^2}$

$\rightarrow f_{xy} = (1 - 6xy - 2x^2 - 6x)e^{y-x^2}$

$\rightarrow f_{yy} = (6 + x + 3y)e^{y-x^2}$

$\rightarrow D = 2.4 \cdot 1.13 - (0.38)^2 = 2.57 > 0$ and $f_{xx} > 0$

$\rightarrow \left(-\frac{1}{6}, -\frac{17}{18}\right)$ is a local minimum

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(29) $f(x,y) = x+y, \quad 0 \le x \le 1, \quad 0 \le y \le 1$

$\rightarrow$ Global minimum: 0

$\rightarrow$ Global maximum: 2

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(35) $f(x,y) = x+y-x^2-y^2-xy, \quad 0 \le x \le 2, \quad 0 \le y \le 2$

$\rightarrow f_x = 1-2x-y$

$\rightarrow f_y = 1-2y-x$

$\rightarrow f_{xx} = -2$

$\rightarrow f_{xy} = -1$

$\rightarrow f_{yy} = -2$

$\rightarrow D = 4-1 = 3 > 0$

$\rightarrow$ Critical Points: $\left(\frac{1}{3}, \frac{1}{3}\right)$

$\rightarrow f\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$

$\rightarrow f_x(x,0) = 1-2x \Rightarrow \left(\frac{1}{2}, 0\right)$

$\rightarrow f\left(\frac{1}{2}, 0\right) = \frac{1}{4}$

$\rightarrow f_y(0,y) = 1-2y \Rightarrow \left(0, \frac{1}{2}\right)$

$\rightarrow f\left(0, \frac{1}{2}\right) = \frac{1}{4}$

$\rightarrow f_y(2,y) = -1-2y \Rightarrow \left(2, -\frac{1}{2}\right)$

$\rightarrow f\left(2, -\frac{1}{2}\right) = -\frac{7}{4}$

$\rightarrow f(x,2) = -1-2x \Rightarrow \left(-\frac{1}{2}, 2\right)$

$\rightarrow f\left(-\frac{1}{2}, 2\right) = -\frac{7}{4}$

$\rightarrow$ The maximum value is $\frac{1}{3}$.
