14.6 Homework

1. a. $\frac{\partial f}{\partial x} = 2xy^3$ $\quad \frac{\partial f}{\partial y} = 3x^2y^2$ $\quad \frac{\partial f}{\partial z} = 4z^3$

b. $\frac{\partial x}{\partial s} = 2s$ $\quad \frac{\partial y}{\partial s} = t^2$ $\quad \frac{\partial z}{\partial s} = 2st$

c. $\frac{\partial f}{\partial s} = (2xy^3)(2s) + (3x^2y^2)(t^2) + (4z^3)(2st)$

$= (2(s)^2(st^2)^3)(2s) + (3(s)^2(st^2)^2)(t^2) + 4(s^2t)^3(2st)$

3. $\frac{\partial f}{\partial s} = (y)(2s) + (x)(2r) + (2z)(0)$

$= 2ys + 2xr = 2(2rs)s + 2(s^2)r = 4rs^2 + 2rs^2 = 6rs^2$

$\frac{\partial f}{\partial r} = (y)(0) + (x)(2s) + (2z)(2r)$

$= 2xs + 4zr \rightarrow 2(s^2)(s) + 4(r^2)(r) = 2s^3 + 4r^3$

5. $\frac{\partial g}{\partial v} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial v}$

$\frac{\partial g}{\partial x} = -\sin(x-y)$ $\quad \frac{\partial x}{\partial v} = 3$ $\quad \frac{\partial g}{\partial y} = \sin(x-y)$ $\quad \frac{\partial y}{\partial v} = -7$

$\frac{\partial g}{\partial u} = -3\sin(x-y) - 7\sin(x-y) = -3\sin(10v-20v) - 7\sin(10v-20v)$

$= -10\sin(10v-20v)$

$\frac{\partial g}{\partial v} = 5\sin(x-y) + 15\sin(x-y) = 20\sin(10v-20v)$

7. $\frac{\partial F}{\partial y}$; $F(u,v) = e^{u+v}$, $u = x^2$ $v = xy$

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial u} \cdot \frac{du}{dy} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= e^{u+v} \cdot 0 + e^{u+v} \cdot x$$
$$= e^{u+v} \cdot x = xe^{x^2+xy}$$

15. $\frac{\partial g}{\partial u}$ at $(u,v) = (0,1)$ $g(x,y) = x^2 - y^2$ $x = e^u \cos v$, $y = e^u \sin v$

$$\frac{\partial g}{\partial u} = \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial g}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= 2x \cdot e^u \cos v + (-2y) e^u \sin v$$
$$= 2(e^u \cos v)^2 - 2(e^u \sin v)^2$$
$$(0,1): \; 2(e^0 \cos(1))^2 - 2(e^0 \sin 1)^2$$
$$2\cos(1)^2 - 2\sin(1)^2$$
$$2(\cos^2(1) - \sin^2(1))$$

23. $x = s+t$ $y = s-t$ $f(x,y)$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot 1 + \frac{\partial f}{\partial y} \cdot 1 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot 1 + \frac{\partial f}{\partial y} \cdot -1 = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial s} \cdot \frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right) \cdot \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right)$$

$$= \left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2$$

27. $\frac{\partial z}{\partial x}$, $x^2y + y^2z + xz^2 = 10$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \qquad F_x = 2xy + z^2$$
$$F_z = y^2 + 2zx$$

$$\frac{\partial z}{\partial x} = \frac{-2xy + z^2}{2xz + y^2}$$

29. $\frac{\partial z}{\partial y}$, $e^{xy} + \sin(xz) + y = 0$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} \qquad F_y = xe^{xy} + 1$$
$$F_z = x\cos(xz)$$

$$\frac{\partial z}{\partial y} = \frac{-xe^{xy} - 1}{x\cos(xz)}$$

35. $\nabla\left(\frac{1}{r}\right)$ Using Eq. 9

Eq. 9 = $F'(r)\, e_r$ where $e_r = \frac{r}{\|r\|}$

$$F'(r) = \frac{-1}{r^2} \qquad \nabla\left(\frac{1}{r}\right) = \frac{-1}{r^2} \cdot \frac{\hat{r}}{\|r\|} = \frac{-\hat{r}}{r^3}$$

14.7 Homework

1. a. $f_x = 2x - 4y$

$2x - 4y = 0 \rightarrow 2(x - 2y) = 0$

$x = 2y \rightarrow a = 2b$

$f_y = 4y^3 - 4x \rightarrow 4(y^3 - x) = 0$

$y^3 = x$ $\quad b^3 = a$

$b^3 = 2b$

$b^3 - 2b = 0$

$b(b^2 - 2) = 0$

$b = 0,\ b = \sqrt{2},\ b = -\sqrt{2}$

so $(0,0)$, $(2\sqrt{2}, \sqrt{2})$, $(-2\sqrt{2}, -\sqrt{2})$

3. $f(x,y) = 8y^4 + x^2 + xy - 3y^2 - y^3$

$f_x = 2x + y$ $\quad f_y = 32y^3 + x - 6y - 3y^2$

$f_{xx} = 2$ $\quad f_{xy} = 1$ $\quad f_{yy} = 96y^2 - 6 - 6y$

$2x + y = 0$ $\quad y = -2x$ $\quad 32(-2x)^3 + x - 6(-2x) - 3(-2x)^2$

$256x^3 + x + 12x - 12x^3 = 0$

$x(256x^2 - 12x + 13) = 0$

$x = 0$ $\quad \dfrac{12 \pm \sqrt{144 - 4(256)(13)}}{512}$

5. $f_x = y^2 - 2yx + y$ $\quad f_y = 2yx - x^2 + x$

$f_{xx} = -2y$ $\quad f_{xy} = 2y - 2x + 1$ $\quad f_{yy} = 2x$

$y(y - 2x + 1) = 0$ $\quad x(2y - x + 1) = 0$

$y = 0$ or $2x + 1$

$x(0 - x + 1) = 0$

$x = 0, 1$

$(0,0)$ $\quad (1,0)$

$f_{xx}(0,0) = 0$ $\quad f_{xy}(0,0) = 1$ $\quad f_{yy}(0,0) = 0$

$-1 \rightarrow$ saddle

$f_{xx}(1,0) = 0$ $\quad f_{xy}(1,0) = -1$ $\quad f_{yy}(1,0) = 2$

$0 - [-1]^2 \quad -1 \rightarrow$ saddle

7. $f(x,y) = x^2+y^2 - xy + x$

$f_x = 2x - y + 1 \qquad f_y = 2y - x$

$f_{xx} = 2 \qquad f_{xy} = -1 \qquad f_{yy} = 2$

$2x - y + 1 = 0 \qquad 2y - x = 0$

$x = 2y$

$2(2y) - y + 1 = 0$

$4y - y + 1 = 0$

$3y = -1 \quad y = -1/3 \qquad 2(-1/3) = \quad x = -2/3$

$(-2/3, -1/3)$

$D = (2)(2) - (-1)^2 = 4 - 1 = 3$ Local min?

11. $f(x,y) = 4x - 3x^3 - 2xy^2$

$f_x = 4 - 9x^2 - 2y^2 \qquad f_y = -4xy$

$f_{xx} = -18x \qquad f_{xy} = -4y \qquad f_{yy} = -4x$

$4 - 9x^2 - 2y^2 = 0 \qquad -4xy = 0$

$x = 0 \quad y = 0$

$4 - 9(0)^2 - 2y^2 = 0$

$4 - 2y^2 = 0$

$y = \pm\sqrt{2}$

$4 - 9x^2 - 0 = 0$

$x = \pm 2/3$

Critical Points: $(0, \sqrt{2})$, $(0, -\sqrt{2})$, $(2/3, 0)$, $(-2/3, 0)$

1 2 3 4

$D_1 = -32$ -> Saddle point

$D_2 = -32$ -> Saddle point

$D_3 = 32$ & $f_{xx} = -12$ -> local max

$D_4 = 32$ & $f_{xx} = 12$ -> local min

13. $f(x,y) = x^4 + y^4 - 4xy$

$f_x = 4x^3 - 4y \qquad f_y = 4y^3 - 4x$

$f_{xx} = 12x^2 \qquad f_{xy} = -4 \qquad f_{yy} = 12y^3$

$4x^3 - 4y = 0 \qquad 4y^3 - 4x = 0$

$4(x^3 - y) \qquad 4(y^3 - x)$

$y = x^3 \qquad 4x^9 - x$

$4x(x^8 - 1) = 0$

$x = 0, \pm 1$

$(0,0)$ 1

$(1,1)$ 2

$(-1,-1)$ 3

$D_1 = -16, \; f_{xx} = \quad \rightarrow$ saddle

$D_2 = 128, \; f_{xx} = 12 \rightarrow$ local min

$D_3 = 128, \; f_{xx} = 12 \rightarrow$ local min

19. $f(x,y) = \ln x + 2\ln y - x - 4y$

$f_x = \frac{1}{x} - 1 \qquad f_y = \frac{2}{y} - 4$

$f_{xx} = -1/x^2 \qquad f_{xy} = 0 \qquad f_{yy} = -2/y^2$

$\frac{1}{x} - 1 = 0 \qquad \frac{2}{y} - 4 = 0$

$x = 1 \qquad 4 = \frac{2}{y} \quad 4y = 2$

$y = 1/2$

$(1, 1/2)$

$D = 8, \; f_{xx} = -1$ so local max

21. $f(x,y) = x - y^2 - \ln(x+y)$

$f_x = 1 - \frac{1}{x+y}$ $\quad f_y = -2y - \frac{1}{x+y}$

$f_{xx} = \frac{1}{(x+y)^2}$ $\quad f_{xy} = \frac{1}{(x+y)^2}$ $\quad f_{yy} = -2 + \frac{1}{(x+y)^2}$

$1 - \frac{1}{x+y} = 0$ $\quad 1 = \frac{1}{x+y}$ $\quad x+y = 1$

I looked up how to solve this and found $(3/2, -1/2)$

$D = -2$ so saddle point

23. $f(x,y) = (x+3y)e^{y-x^2}$

$f_x = (x+3y)(e^{y-x^2} \cdot -2x) + (1)(e^{y-x^2})$

$= (1 - 2x^2 - 6xy)e^{y-x^2}$

$f_y = (3+x+3y)e^{y-x^2}$

$(1 - 2x^2 - 6xy) = 0$ $\quad$ Using calculator, I got

$3 + x + 3y = 0$ $\quad (-1/6, -17/18)$

Then $D = (2.4)(0.41) - (1.13)^2 = 2.57 > 0$ and $2.4 > 0$

so local min

29. $f(x,y) = x+y$ $\quad 0 \le x \le 1$ $\quad 0 \le y \le 1$

$f(0,0) = 0+0 = 0 \rightarrow$ Max

$f(1,1) = 1+1 = 2 \Rightarrow$ minimum

35. Maximum of: $f(x,y) = x+y-x^2-y^2-xy$

$0 \le x \le 2 \quad 0 \le y \le 2$

$f_x = 1-2x-y \quad f_y = 1-2y-x$

$f_{xx} = -2 \quad f_{xy} = -1 \quad f_{yy} = -2$

$1-2x-y=0 \quad 1-2y-x=0$

$y = 1-2x \quad 1-2(1-2x)-x=0$

$y = 1-2(1/3) \quad 1-2+4x-x=0$

$y = 1-2/3 \quad -1+3x=0$

$y = 1/3 \quad x = 1/3$

$(1/3, 1/3)$

$f(1/3,1/3) = 1/3+1/3-(1/3)^2-(1/3)^2-(1/3)(1/3)$

$= 2/3 - 1/9 - 1/9 - 1/9 = 2/3 - 3/9$

$6/9 - 3/9 = \boxed{3/9} = 1/3$

Left side: $x=0 \quad y = 0 \le y \le 2$

F $f(0,y) = y - y^2$

$f'(0,y) = 1-2y$

$1-2y=0 \quad y = 1/2 \quad F(1/2)$

$(1/2)-(1/2)^2 =$

$1/2 - 1/4 = \boxed{1/4}$

$F(0) = \boxed{0}$

$F(2) = \boxed{-2}$

Right Side: $x=2 \quad y = 0 \le y \le 2$

$f(2,y) = 2+y-4-y^2-2y = -2+y-y^2-2y = F$

$F' = 1-2y-2$

$1-2y-2=0$

$y = 1/2 \quad F(1/2) = -2+1/2-1/4-2$

$-6/4 - 2 = -15/4$

Up: Same thing but $y=0$

Down: Same thing but $y=2$

Compare all the contenders!