14.6, 14.7 HW 10/11/20

14.6 : 1, 3, 5, 7, 15, 17, 23, 27, 29, 31

1. Let $f(x,y,z) = x^2y^3 + z^4$ & $x = s^2$, $y = st^2$, & $z = s^2t$

(a) $df/dx = 2xy^3$; $df/dy = 3x^2y^2$; $df/dz = 4z^3$

(b) $dx/ds = 2s$; $dy/ds = t^2$; $dz/ds = 2st$

(c) $\frac{df}{ds} = \frac{df}{dx} \cdot \frac{dx}{ds} + \frac{df}{dy} \cdot \frac{dy}{ds} + \frac{df}{dz} \cdot \frac{dz}{ds} = 2xy^3 \cdot 2s + 3x^2y^2t^2 + 4z^3 \cdot 2st$

$= 4xy^3s + 3x^2y^2t^2 + 8z^3st$

$df/ds = 4s^2(st^2)^3 s + 3(s^2)^2(st^2)^2t^2 + 8(s^2t)^3 st = 4s^6t^6 + 3s^6t^6 + 8s^7t^4$

$= 7s^6t^6 + 8s^7t^4$

In Exs. 3, 5, 7, use the Chain Rule to calc. the partial deriv. Express the answer in terms of the independent vars.

3. df/ds, df/dr ; $f(x,y,z) = xy + z^2$, $x = s^2$, $y = 2rs$, $z = r^2$

$df/dx = y$, $df/dy = x$, $df/dz = 2z$

$\frac{df}{ds} = \frac{df}{dx} \cdot \frac{dx}{ds} + \frac{df}{dy} \cdot \frac{dy}{ds} + \frac{df}{dz} \cdot \frac{dz}{ds}$ | $\frac{dx}{ds} = 2s$, $\frac{dy}{ds} = 2r$, $\frac{dz}{ds} = 0$

$\frac{df}{dr} = \frac{df}{dx} \cdot \frac{dx}{dr} + \frac{df}{dy} \cdot \frac{dy}{dr} + \frac{df}{dz} \cdot \frac{dz}{dr}$ | $\frac{dx}{dr} = 0$, $\frac{dy}{dr} = 2s$, $\frac{dz}{dr} = 2r$

$df/ds = y \cdot 2s + x \cdot 2r + 2z \cdot 0 = 2ys + 2xr$

$df/dr = y \cdot 0 + x \cdot 2s + 2z \cdot 2r = 2xs + 4zr$

$df/ds = 2rs \cdot 2s + s^2 \cdot 2r = 4rs^2 + 2rs^2 = 6rs^2$

6. $df/dr = 2s^2 \cdot s + 4r^2 \cdot r = 2s^3 + 4r^3$

dg/du, dg/dv ; $R(x,y) = (3x+4y)^5$, $x = u^2$, $y = uv$

$dR/dx = 5(3x+4y)^4 \cdot 3 = 15(3x+4y)^4$, $dR/dy = 5(3x+4y)^4 \cdot 4 = 20(3x+4y)^4$

$\frac{dg}{du} = \frac{dR}{dx} \cdot \frac{dx}{du} + \frac{dR}{dy} \cdot \frac{dy}{du}$ | $\frac{dx}{du} = 2u$ $\frac{dy}{du} = v$

$\frac{dg}{dv} = \frac{dR}{dx} \cdot \frac{dx}{dv} + \frac{dR}{dy} \cdot \frac{dy}{dv}$ | $\frac{dx}{dv} = 0$ $\frac{dy}{dv} = u$

$dg/du = 5(3x+4y)^4 \cdot 2u + 20(3x+4y)^4 \cdot v = (3x+4y)^4(10u + 20v)$

$dg/dv = 5(3x+4y)^4 \cdot 0 + 20(3x+4y)^4 \cdot u = 20u(3x+4y)^4$

$dg/du = (3u^2 + 4uv)^4(10u + 20v)$

$dg/dv = 20u(3u^2 + 4uv)^4$

7. dF/dy ; $F(u+v) = e^{u+v}$, $u = x^2$, $v = xy$

$df/du = e^{u+v}$, $df/dv = e^{u+v}$

$$\frac{dF}{dy} = \frac{dF}{du}\cdot\frac{du}{dy} + \frac{dF}{dv}\cdot\frac{dv}{dy} = e^{u+v}\frac{du}{dy} + e^{u+v}\frac{dv}{dy} = e^{u+v}\left(\frac{du}{dy} + \frac{dv}{dy}\right)$$

$du/dy = 0$, $dv/dy = x$; $dF/dy = e^{u+v}(0+x) = xe^{u+v}$

$dF/dy = xe^{x^2+xy}$

dg/du at $(u,v) = (0,1)$, where $g(x,y) = x^2 - y^2$, $x = e^u \cos v$, $y = e^u \sin v$

$$dg/dx = 2x,\ dg/dy = -2y \Rightarrow \frac{dg}{du} = \frac{dg}{dx}\cdot\frac{dx}{du} + \frac{dg}{dy}\cdot\frac{dv}{du} = 2x\frac{dx}{du} - 2y\frac{dy}{du}$$

$dx/du = e^u \cos v$, $dy/du = e^u \sin v$

$\cos^2 x - \sin^2 x = \cos 2x$

$dg/du = 2xe^u\cos v - 2ye^u \sin v = 2e^u(x\cos v - y\sin v)$

$x = e^0 \cos 1 = \cos 1$, $y = e^0 \sin 1 = \sin 1$

$dg/du \,|_{(u,v)=(0,1)} = 2e^0(\cos^2 1 - \sin^2 1) = 2\cdot\cos 2\cdot 1 = 2\cos 2$

Dist. between 2 players : $D(t) = \sqrt{x(t)^2 + y(t)^2} \Rightarrow \frac{dD}{dt} = \frac{dD}{dx}\cdot\frac{dx}{dt} + \frac{dD}{dy}\cdot\frac{dy}{dt}$

Given: $dx/dt = 20$ & $dy/dt = 18$

$$\frac{dD}{dx} = \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{dD}{dy} = \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2y = \frac{y}{\sqrt{x^2+y^2}}$$

The dist. between th[e] players is decr. at a rate of 26.8 ft/se[c]

$dD/dx\,|_{(x,y)=(-8,-6)} = -8/\sqrt{8^2+6^2} = -4/5$

$dD/dx\,|_{(x,y)=(-8,-6)} = -6/\sqrt{8^2+6^2} = -3/5$

$dD/dt = -4/5 \cdot 20 - 3/5 \cdot 18 = -134/5 = -26.8$ ft/sec

23. Let $x = s+t$ & $y = s-t$. Show that for any diff. fn. $f(x,y)$,

$(df/dx)^2 - (df/dy)^2 = df/ds \cdot df/dt$

$$\frac{df}{ds} = \frac{df}{dx}\cdot\frac{dx}{ds} + \frac{df}{dy}\cdot\frac{dy}{ds} = \frac{df}{dx}\cdot 1 + \frac{df}{dy}\cdot 1 = \frac{df}{dx} + \frac{df}{dy}$$

$$\frac{df}{dt} = \frac{df}{dx}\cdot\frac{dx}{dt} + \frac{df}{dy}\cdot\frac{dy}{dt} = \frac{df}{dx}\cdot 1 + \frac{df}{dy}\cdot(-1) = \frac{df}{dx} - \frac{df}{dy}$$

$$\frac{df}{ds}\cdot\frac{df}{dt} = \left(\frac{df}{dx} + \frac{df}{dy}\right)\cdot\left(\frac{df}{dx} - \frac{df}{dy}\right) = \left(\frac{df}{dx}\right)^2 - \left(\frac{df}{dy}\right)^2$$

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In Exs. 27, 29, 31, calc. partial deriv. using implicit diff.

27. dz/dx, $x^2y + y^2z + xz^2 = 10$

$dz/dx = -F_x/F_z \Rightarrow F_x = 2xy + z^2$, $F_z = y^2 + 2xz$

$dz/dz = -(2xy + z^2)/(2xz + y^2)$

29. dz/dy, $e^{xy} + \sin(xz) + y = 0 \Rightarrow dz/dy = -F_y/F_z$

$F_y = xe^{xy} + 1$, $F_z = x\cos(xz) \Rightarrow dz/dy = -(xe^{xy}+1)/(x\cos(xz))$

31. $\dfrac{dw}{dy}$, $\dfrac{1}{w^2+x^2} + \dfrac{1}{w^2+y^2} = 1$ at $(x, y, w) = (1, 1, 1) \Rightarrow \dfrac{dw}{dy} = -\dfrac{F_y}{F_w}$

$F_y = -\dfrac{2y}{(w^2+y^2)^2}$, $F_w = \dfrac{-2w}{(w^2+x^2)^2} - \dfrac{2w}{(w^2+y^2)^2}$

$$\frac{dw}{dy} = -\frac{-2y/(w^2+y^2)^2}{-2w/(w^2+y^2)^2 - 2w/(w^2+y^2)^2} = -\frac{y(w^2+x^2)^2}{w(w^2+y^2)^2 + w(w^2+x^2)^2}$$

$$= \frac{-y(w^2+x^2)^2}{w((w^2+y^2)^2 + (w^2+x^2)^2)}$$

<u>14.7</u>: 1, 3, 5, 7, 11, 13, 17, 19, 23, 29, 35

1. Let $P = (a, b)$ be a crit. pt. of $f(x,y) = x^2 + y^4 - 4xy$

(a) First use $f_x(x,y) = 0$ to show that $a = 2b$. Then use $f_y(x,y) = 0$ to show that $P = (0,0)$, $(2\sqrt{2}, \sqrt{2})$, or $(-2\sqrt{2}, -\sqrt{2})$

$f_x(x,y) = d/dx\,(x^2 + y^4 - 4xy) = 2x - 4y$

$f_y(x,y) = d/dy\,(x^2 + y^4 - 4xy) = 4y^3 - 4x$

$2a - 4b = 0 \Rightarrow a = 2b$ / $4b^3 - 4a = 0 \Rightarrow a = b^3$

$2b = b^3 \Rightarrow b^3 - 2b = b(b^2 - 2) = 0 \Rightarrow \{b_1 = 0, b_2 = \sqrt{2}, b_3 = -\sqrt{2}$

Since $a = 2b$; $a_1 = 0$, $a_2 = 2\sqrt{2}$, $a_3 = -2\sqrt{2}$ crit. pts.

$P_1 = (0, 0)$, $P_2 = (2\sqrt{2}, \sqrt{2})$, $P_3 = (-2\sqrt{2}, -\sqrt{2})$

(b) Referring to Fig. 18, we see that $P_1 = (0,0)$ is a saddle pt. & $P_2 = (2\sqrt{2}, \sqrt{2})$, $P_3 = (-2\sqrt{2}, -\sqrt{2})$ are local minima. The abs. min. val. of f is -4.

3. Find the crit. pts. of $f(x,y) = 8y^4 + x^2 + xy - 3y^2 - y^3$. Use the contour map in Fig. 20 to determine their nature (local min., local max., or saddle pt.).

Crit pts. are the sols. of $f_x = 0$ & $f_y = 0$

$f_x(x,y) = 2x + y = 0 \Rightarrow y = -2x$

$f_y(x,y) = 32y^3 + x - 6y - 3y^2 = 0$

$32(-2x)^3 + x - 6(-2x) - 3(-2x)^2 = 0$

$-256x^3 + 13x - 12x^2 = 0$

$-x(256x^2 + 12x - 13) = 0 \Rightarrow \overset{①}{x=0}$ or $\overset{②,③}{256x^2 + 12x - 13 = 0}$

$x_{1,2} = \dfrac{-12 \pm \sqrt{12^2 - 4 \cdot 256 \cdot (-13)}}{512} = \dfrac{-12 \pm 116}{512} \Rightarrow x = \dfrac{13}{64}$ ② or $-\dfrac{1}{4}$ ③

Sub. in $y = -2x$ (y-coords of crit. pts.): $(0,0)$, $(13/64, -13/32)$, $(-1/4, 1/2)$

Contour map: level curves through $(0,0)$ consist of 2 intersecting lines that divide the neighborhood near $(0,0)$ into 4 regions. f is decr. in y-dir. & incr. in x-dir. $\Rightarrow (0,0)$ is saddle pt. Level curves near crit. pts. $(13/64, -13/32)$ & $(-1/4, 1/2)$ are closed curves encircling the pts, hence these are local min. or max. $\Rightarrow$ graph shows that both crit. pts. are local mins.

5. Let $f(x,y) = y^2x - yx^2 + xy$

(a) Show that the crit. pts. (x,y) satisfy the eqs. $y(y-2x+1) = 0$ & $x(2y - x + 1) = 0$

$f_x(x,y) = y^2 - 2yx + y = 0 \Rightarrow y(y - 2x + 1) = 0$ (1)

$f_y(x,y) = 2yx - x^2 + x = 0 \qquad x(2y - x + 1) = 0$ (2)

(b) Find crit. pts. by solving (a) eqs.

(1) $\Rightarrow y = 0$ or $y = 2x - 1$ // Sub $y = 0$ in (2)

$x(-x+1) = 0 \Rightarrow x = 0$ or $x = 1 \Rightarrow (0,0)$ & $(1,0)$ Sols.

Now sub. $y = 2x - 1$ in (2)

$x(4x - 2 - x + 1) = 0 \Rightarrow x(3x - 1) = 0 \Rightarrow x = 0$ or $x = 1/3$

Compute y-coord. using $y = 2x - 1$:

$y = 2 \cdot 0 - 1 = -1$ // $y = 2 \cdot 1/3 - 1 = -1/3 \Rightarrow (0,-1)$ & $(1/3, -$

(c) 2nd order partial derivs: $f_{xx}(x,y) = d/dx\,(y^2 - 2yx + y) = -2y$

$f_{yy}(x,y) = d/dy\,(2yx - x^2 + x) = 2x$

$f_{xy}(x,y) = d/dy\,(y^2 - 2yx + y) = 2y - 2x + 1$

Discriminant: $D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = -2y \cdot 2x - (2y - 2x + 1)^2$

$= -4xy - (2y - 2x + 1)^2$

Step 1. Find crit. pts. (set 1^{st} order derivs. $=0$)

Step 2. Compute discriminant (2nd order partials)

Step 3. 2nd Deriv. Test

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2nd Deriv. Test $\Rightarrow$ 1^{st} compute discriminants at crit. pts:

$D(0,0) = -1 < 0$

$D(1,0) = -1 < 0$

$D(0,-1) = -1 < 0$

$D(1/3, -1/3) = -4 \cdot 1/3(-1/3) - (-2/3 - 2/3 + 1)^2 = 1/3 > 0$

$f_{xx}(1/3, -1/3) = -2 \cdot (-1/3) = 2/3 > 0$

$(0,0)$, $(1,0)$, & $(0,-1)$ are saddle pts. & $f(1/3, -1/3)$ is a local min.

For 7, 11, 13, 17, 19, & 23, find crit. pts. of f. Then use 2nd Deriv. Test to determine whether they are local min., local max., or saddle pts.

7. $f(x,y) = x^2 + y^2 - xy + x$

S1. $f_x(x,y) = 2x - y + 1 = 0$ (1) sub.

$f_y(x,y) = 2y - x = 0$ (2) $\Rightarrow x = 2y$

$2 \cdot 2y - y + 1 = 0 \Rightarrow 3y = -1 \Rightarrow y = -1/3$

$x = 2 \cdot (-1/3) = -2/3 \Rightarrow$ Crit. pt.: $(-2/3, -1/3)$

S2. $f_{xx}(x,y) = 2$, $f_{yy}(x,y) = 2$, $f_{xy}(x,y) = -1$

$D(x,y) = f_{xx} f_{yy} - f_{xy}^2 = 2 \cdot 2 - (-1)^2 = 3$

S3. $D(-2/3, -1/3) = 3 > 0$ & $f_{xx}(-2/3, -1/3) = 2 > 0$

$f(-2/3, -1/3)$ is a local min.

11. $f(x,y) = 4x - 3x^3 - 2xy^2$

S1. $f_x(x,y) = 4 - 9x^2 - 2y^2 = 0$ (1) sub.

$f_y(x,y) = -4xy = 0$ (2) $\Rightarrow x = 0$ or $y = 0$

$4 - 2y^2 = 0 \Rightarrow y^2 = 2 \Rightarrow y = \sqrt{2}$, $y = -\sqrt{2}$

$4 - 9x^2 = 0 \Rightarrow 9x^2 = 4 \Rightarrow x = 2/3$, $x = -2/3$

Crit. pts.: $(0, \sqrt{2})$, $(0, -\sqrt{2})$, $(2/3, 0)$, $(-2/3, 0)$

S2. $f_{xx}(x,y) = -18x$, $f_{yy}(x,y) = -4x$, $f_{xy} = -4y$

$D(x,y) = f_{xx} f_{yy} - f_{xy}^2 = -18x \cdot (-4x) - (-4y)^2 = 72x^2 - 16y^2$

S3. $D(0, \sqrt{2}) = -32 < 0$

$D(0, -\sqrt{2}) = -32 < 0$

$D(2/3, 0) = 72 \cdot 4/9 = 32 > 0$

$f_{xx}(2/3, 0) = -18 \cdot 2/3 = -12 < 0$

$D(-2/3, 0) = 72 \cdot 4/9 = 32 > 0$

$f_{xx}(-2/3, 0) = -18 \cdot (-2/3) = 12 > 0$

$(0, \pm\sqrt{2})$ are saddle pts., $f(2/3, 0)$ local max., $f(-2/3, 0)$ local min.

13. $f(x,y) = x^4 + y^4 - 4xy$

S1. $f_x(x,y) = 4x^3 - 4y = 0$ (1) $\Rightarrow y = x^3$ sub.

$f_y(x,y) = 4y^3 - 4x = 0$ (2)

$(x^3)^3 - x = x^9 - x = x(x^8 - 1) = 0 \Rightarrow x = 0$, $x = 1$, $x = -1$

$y = 0^3 = 0$, $y = 1^3 = 1$, $y = (-1)^3 = -1$

Crit. pts: $(0,0)$, $(1,1)$, $(-1,-1)$

S2. $f_{xx}(x,y) = 12x^2$, $f_{yy}(x,y) = 12y^2$, $f_{xy}(x,y) = -4$

$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 12x^2 \cdot 12y^2 - (-4)^2 = 144x^2y^2 - 16$

S3. $D(0,0) = -16 < 0$ | $f_{xx}(1,1) = 12 > 0$

$D(1,1) = 144 - 16 = 128 > 0$ | $f_{xx}(-1,-1) = 12 > 0$

$D(-1,-1) = 144 - 16 = 128 > 0$

$(0,0)$ is a saddle pt. // $f(1,1)$ & $f(-1,-1)$ are local min.

$f(x,y) = \sin(x+y) - \cos x$

S1. $f_x(x,y) = \cos(x+y) + \sin x = 0$ (1) $\Rightarrow \cos(x+y) = 0$

$f_y(x,y) = \cos(x+y) = 0$ (2)

(1) $x + y = \frac{(2K+1)\pi}{2} \Rightarrow y = \frac{(2K+1)\pi}{2} - x$ where K is an int.

SKIP

$x = K\pi$ & $y = ((2n+1)\pi)/2$ where n, K are ints.

S2. $f_{xx}(x,y) = -\sin(x+y) + \cos x$, $f_{yy}(x,y) = -\sin(x+y)$, $f_{xy}(x,y) = -\sin(x+y)$

$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = (-\sin(x+y) + \cos x)(-\sin(x+y)) - \sin^2(x+y)$

$= -\cos(x)\sin(x+y)$

S3. $D = \begin{cases} +1, & \text{if } y = ((4n+3)\pi)/2 \\ -1, & y = ((4n+1)\pi)/2 \end{cases}$

$(K\pi, ((4n+1)\pi)/2)$ are saddle pts. since $D < 0$

Since $D > 0$ for the pts. $(K\pi, ((4n+3)\pi/2)$, we need to examine f_{xx}

$f_{xx} > 0$ if K is even & $f_{xx} < 0$ if K is odd

$(K\pi, ((4n+3)\pi)/2)$ are local min. if K is even

$(K\pi, ((4n+3)\pi)/2)$ are local max. if K is odd

$f(x,y) = \ln x + 2\ln y - x - 4y$

S1. $f_x(x,y) = 1/x - 1 = 0$ (1) $\Rightarrow x = 1$ $\Rightarrow$ Crit. pt: $(1, 1/2)$

$f_y(x,y) = 2/y - 4 = 0$ (2) $\Rightarrow y = 1/2$

S2. $f_{xx}(x,y) = -1/x^2$, $f_{yy}(x,y) = -2/y^2$, $f_{xy}(x,y) = 0$

Do S1-S3 again! $D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 2/x^2y^2$

S3. $f(1, 1/2)$ is a local max.

$f(x,y) = (x+3y)e^{y-x^2}$

$f_x(x,y) = 1 \cdot e^{y-x^2} + (x+3y)e^{y-x^2} \cdot (-2x) = e^{y-x^2}(1 - 2x^2 - 6xy)$

$f_y(x,y) = 3e^{y-x^2} + (x+3y)e^{y-x^2} \cdot 1 = e^{y-x^2}(3 + x + 3y)$

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SKIP

$e^{y-x^2}(1-2x^2-6xy)=0 \Rightarrow 1-2x^2-6xy=0$ (Since $e^{y-x^2}\neq 0$)

$e^{y-x^2}(3+x+3y)=0 \qquad 3+x+3y=0$

Crit. pt: $(-1/6, -17/18)$

S2. $f_{xx}(x,y) = 2e^{y-x^2}(2x^3+6x^2y-3x-3y)$ | $D(x,y) = f_{xx}f_{yy}-f_{xy}^2$

$f_{yy}(x,y) = e^{y-x^2}(6+x+3y)$

$f_{xy}(x,y) = e^{y-x^2}(1-6xy-2x^2-6x)$

S3.

Crit. Pt.	f_{xx}	f_{yy}	f_{xy}	D	Type
$(-1/6, -17/18)$	2.4	1.13	0.38	2.57	$D>0$, $f_{xx}>0$, local min.

$f(x,y)=x+y$, $0\le x\le 1$, $0\le y\le 1$

Sum of $x+y$ is max when $x=1$ & $y=1$, & it is min when $x=0$ & $y=0$. Global max of f: $f(1,1)=1+1=2$ & global min of f: $f(0,0)=0+0=0$

SKIP

Find the max. of $f(x,y)=x+y-x^2-y^2-xy$ on the □, $0\le x\le 2$, $0\le y\le 2$ (Fig. 23)

(a) $f_x(x,y)=1-2x-y$, $f_y(x,y)=1-2y-x$

$f(1/3, 1/3) = 1/3 + 1/3 - 1/9 - 1/9 - 1/9 = 1/3$

(b) $f'(x,0) = 1-2x \Rightarrow 0$ at $x=1/2$

$\underbrace{f(1/2)=1/4}_{\text{max.}}$, $\underbrace{f(0,0)=0, \quad f(2,0)=0}_{\text{min.}}$

(c) left: max $=1/4$, min $=0$

top: max $=-2$, min $=-8$

right: max $=-2$, min $=-8$

(d) Largest val. is $1/3$, achieved at $(x,y)=(1/3,1/3)$. This is the max. val. of f on the given □.