# 14.6

1. Let $f(x,y,z) = x^2y^3 + z^4$

$x = s^2$ $\quad y = st^2$ $\quad z = s^2t$

$\frac{df}{dx} = 2xy^3$ $\qquad \frac{dx}{ds} = 2s$

$\frac{df}{dy} = 3x^2y^2$ $\qquad \frac{dy}{ds} = t^2$

$\frac{df}{dz} = 4z^3$ $\qquad \frac{dz}{ds} = 2st$

$$\frac{df}{ds} = 4xy^3s + 3x^2y^2t^2 + 8z^3st$$

3) $f(x,y,z) = xy + z^2$

$x = s^2$ , $y = 2rs$ $z = r^2$

$\frac{df}{dx} = y$ $\frac{dx}{dr} = 0$

$\frac{df}{dy} = x$ $\frac{dy}{dr} = 2s$

$\frac{df}{dz} = 2z$ $\frac{dz}{dr} = 2r$

$\frac{df}{dr} = 2xs + 4zr$ $\left\{ \text{independent variables?} \right\}$

5) $g(x,y) = \cos(x-y)$

, $x = 3u - 5v$ , $y = -7u + 15v$

$g_x = -\sin(x-y)$ $\quad x_u = 3$

$g_y = \sin(x-y)$ $\quad y_u = -7$

$$\frac{dg}{du} = -\sin(x-y)(3+7)$$

$$= -10\sin(x-y)$$

7) $\frac{dF}{dy} = ?$

$F(u,v) = e^{u+v}$

$u = x^2$ $\quad v = xy$

$F_u = e^{u+v}$

$u_y = 0$

$F_v = e^{u+v}$

$v_y = x$

$\frac{df}{dy} = \quad x e^{u+v}$

15) $\frac{dg}{du}$ at $(u,v) = (0,1)$

$g(x,y) = x^2 - y^2$

$x = e^u \cos v \qquad y = e^u \sin v$

$g_x = 2x \qquad \frac{dx}{du} = e^u \cos v = 1$

$g_y = -2y \qquad \frac{dy}{du} = e^u \sin v = 0$

$$\frac{dg}{du} = 2x$$

17) $\rightarrow$ optional

23) $x = s+t \qquad y = s-t$

Let $f(x,y) = x^2 + y^2$

$$\frac{df}{dx} = 2x \qquad \frac{dx}{ds} = 1 \qquad \frac{dx}{dt} = 1$$

$$\frac{df}{dy} = 2y \qquad \frac{dy}{ds} = 1 \qquad \frac{dy}{dt} = -1$$

$$\frac{df}{ds} = 2x + 2y$$

$$\frac{df}{dt} = 2x - 2y$$

$$\Rightarrow \frac{df}{ds} \cdot \frac{df}{dt} = (2x+2y)(2x-2y)$$

RHS $\Rightarrow$ $4x^2 - 4y^2$

$$\Rightarrow \left(\frac{df}{dx}\right)^2 = 4x^2$$

$$\Rightarrow \left(\frac{df}{dy}\right)^2 = 4y^2$$

LHS = $4x^2 - 4y^2$

$\therefore$ LHS = RHS

Hence proved

27) $x^2y + y^2z + xz^2 = 10$

$\dfrac{dz}{dx} = ?$

$2xy + y^2\dfrac{dz}{dx} + z^2 + 2xz\dfrac{dz}{dx}$

$2xy + z^2 = (-y^2 - 2xz)\dfrac{dz}{dx}$

$\therefore \dfrac{dz}{dx} = -\dfrac{(2xy + z^2)}{(2xz + y^2)}$

29) $\dfrac{dz}{dy} = ?$ $\quad e^{xy} + \sin(xz) + y = 0$

$x.e^{xy} + \cos(xz)\left(x.\dfrac{dz}{dx}\right) + 1 = 0$

$\dfrac{dz}{dx} = \dfrac{-(xe^{xy} + 1)}{x\cos xz}$

31) $\dfrac{1}{(w^2+x^2)'} + \dfrac{1}{w^2+y^2} = 1$

$(x, y, w) = (1, 1, 1)$ . $\dfrac{dw}{dy}$

$$-(w^2+x^2)^{-2}\left(2w\frac{dw}{dy}\right) - (w^2+y^2)^{-2}\left(2w\frac{dw}{dy}+2y\right)$$

$$-\frac{1}{2\cancel{2}}\left(\cancel{2}\frac{dw}{dy}\right) - \frac{1}{\cancel{4}_2}\left(\cancel{2}\frac{dw}{dy}+\cancel{2}\right)$$

$$\frac{dw}{dy} + \frac{dw}{dy} + 1 = 0$$

$$\frac{dw}{dy} = -\frac{1}{2}$$

# 14.7

1) $P=(a,b)$

$f(x,y) = x^2+y^4-4xy$

(a) $f_x = 2x-4y = 0$

$\cancel{2}x = \cancel{4}^{2}y$

$x = 2y$

Here $x=a$ & $y=b$

$\therefore\ a=2b.$

$f_y = 4y^3-4x = 0$

$y^3 - x = 0$

$y^3-2y = 0$

$$y(y^2-2)=0$$

$$\boxed{\begin{array}{l} y=0, \quad y=\sqrt{2}, \quad y=-\sqrt{2} \\ x=0, \quad x=2\sqrt{2}, \quad x=-2\sqrt{2} \end{array}}$$

(b) $f_{xy} = -4$

$f_{yy} = 12y^2$

$f_{xx} = 2$ $\quad > 0$

$D = 24y^2 - 16$

At $D_{(0,0)} = -16 \quad < 0$

$D_{(2\sqrt{2},\sqrt{2})} = 24(2)-16 = 32 > 0$

$D_{(-2\sqrt{2},-\sqrt{2})} = 24(2)-16 = 32 > 0$

At pt. $(0, 0)$

$D < 0$, $\therefore$ $(0, 0)$ saddle point

At pt $(2\sqrt{2}, \sqrt{2})$ and $(-2\sqrt{2}, -\sqrt{2})$

$D > 0$, $f_{xx} > 0$

$\therefore$ these are local min.

local min. values

$f(x, y) = x^2 + y^4 - 4xy$

$f(2\sqrt{2}, \sqrt{2}) = 8 + 4 - 16$

$= -4$

$f(-2\sqrt{2}, -\sqrt{2}) = 8 + 4 + 16$

$= 28$

3)

$f(x,y) = 8y^4 + x^2 + xy - 3y^2 - y^3$

$f_x = 2x + y$

$f_y = 32y^3 + x - 6y - 3y^2$

$f_{xx} = 2$

$f_{xy} = 1$

$f_{yy} = 96y^2 - 6 - 6y$

$f_x = 0$ | $f_y = 0$

$2x + y = 0$ | $32y^3 - \frac{y}{2} - 6y - 3y^2 = 0$

$x = -y/2$ | $y = 0,\ 0.5,\ \frac{13}{32}$

$x = 0,\ -\frac{1}{4},\ \frac{13}{64}$

| At $(0,0)$ | $(1/2, -1/4)$ | $(13/32, 13/64)$ |
|---|---|---|
| $f_{yy} f_{xx} - (f_{xy})^2$ | | |
| $D = \; < 0$ | , $0$ | , $0$ |
| saddle point | inconclusive | inconclusive |

5) $f(x,y) = y^2x - yx^2 + xy$

$f_x = y^2 - 2xy + y$

$y(y - 2x + 1) = 0$

$f_y = 2xy - x^2 + x$

$x(2y - x + 1) = 0$

On solving $f_x, f_y = 0$

Critical point $= \left(\frac{1}{3}, -\frac{1}{3}\right) \; (0, -1)$

$f_{xx} = -2y$

$f_{xy} = 2y - 2x + 1$

$f_{yy} = 2x$

At pt $(0,-1)$

$f_{xx} = 2$

$f_{xy} = -1$

$f_{yy} = 0$

$D = 1$

$\therefore (0,-1)$ a local min

At pt. $\left(\frac{1}{3}, -\frac{1}{3}\right)$

$f_{xx} = 2/3 > 0$

$f_{xy} = \left(-\frac{4}{3} + 1\right) = -\frac{1}{3}$

$f_{yy} = -2/3$

$D = \frac{4}{9} - \frac{1}{9} = \frac{3}{9} > 0$

$\therefore \left(\frac{1}{3}, -\frac{1}{3}\right)$ a local min

7) $f(x,y) = x^2 + y^2 - xy + 8$

$x = y/2$

$f_x = 2x - y$

$2y - y/2 = 0$

$f_y = 2y - x$

$4y - y = 0$

$f_{xx} = 2 > 0$

$3y = 0$

$f_{xy} = -1$

Critical pt $\rightarrow (0,0)$

$f_{yy} = 2$

$D = 16 - 1$
$= 15 > 0$

Since $D > 0$ & $f_{xx} > 0$

$(0,0)$ is a local min at 8

11) $f(x,y) = 4x - 3x^3 - 2xy^2$

$f_x = 4 - 9x^2 - 2y^2$

$f_y = -4xy$

Critical point $= (0,0)$

$f_{xx} = -18x$

$f_{yy} = -4x$

$f_{xy} = -4y$

$D = 72x^2 - 16y^2$

$= 0$

unconclusive test

23) $f(x,y) = (x+3y)e^{y-x^2}$

$f(x,y) = xe^{y-x^2} + 3ye^{y-x^2}$

$f_x = e^{y-x^2} + xe^{y-x^2}(-2x) +$

$3ye^{y-x^2}(-2x)$

$f_x = e^{y-x^2}(1-2x^2-6xy)$

$-2x^2 - 6xy + 1 = 0$

$f_y = xe^{y-x^2} + 3ye^{y-x^2} + 3e^{y-x^2}$

$= e^{y-x^2}(x+3y+3)$

$x+3y+3 = 0$

$x, y = -\frac{1}{6}, -\frac{17}{18}$

$$f_{xx} = (-4x - 6y) = \frac{\cancel{4}^{2}}{\cancel{6}_{3}} + \cancel{6}^{1}\left(\frac{17}{\cancel{18}_{3}}\right)$$

$$f_{xy} = (-6x) = 1 \qquad = \frac{19}{3} > 0$$

$$f_{yy} = (3)$$

$$D = 19^2 - 1$$
$$= 360 \qquad > 0$$

Since $D > 0$ & $f_{xx} > 0$

its a local min.

29) $f(x,y) = x+y$

$x \in [0,1]$ , $y \in [0,1]$

| | |
|---|---|
| $x=0,\ 0 \le y \le 1$ | $x=1,\ 0 \le y \le 1$ |
| $f(0,y) = y$ | $f(1,y) = 1+y$ |
| $F(y) = y$ | $F(y) = 1+y$ |
| $F'(y) = 1$ | $F'(y) = 1$ |
| $y = 0, 1$ | $y = (1,2)$ |

| | |
|---|---|
| $y=0 \quad 0 \le x \le 1$ | $y=1 \quad 0 \le x \le 1$ |
| $f(x,0) = x = F(x)$ | $y = 0 \le x \le 1$ |
| $F'(x) = 1$ | $f(x,1) = x+1$ |
| $x = 0, 1$ | $= F(x)$ |

$F(x) = 1$

local min $= 0$

local max $= 2$

35) $f(x,y) = x + y - x^2 - y^2 - xy$

$0 \le x \le 2, \quad 0 \le y \le 2$

$f_x = 1 - 2x - y$ $\qquad f_x = 0$

$f_y = 1 - 2y - x$ $\qquad f_y = 0$

On solving

$(x,y) \to \left(\frac{1}{3}, \frac{1}{3}\right)$

$f\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{2}{3} - \frac{1}{9} - \frac{1}{9} - \frac{1}{9}$

$= \frac{2}{3} - \frac{1}{3}$

$= 1/3$

$x = 0, \quad 0 \leq y \leq 2$

$f(0, y) = y - y^2$

$F'y = -2y$

At $y = 0$

$Fy = 0$

$y = 2$

$Fy = -2$

$x = 2, \quad 0 \leq y \leq 2$

$f(2, y) = y - y^2 - 2y + 2$

$-4$

$= y - y^2 - 2y - 2$

$F'y = -2y - 2$

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