15.6 & 16.1 (Nov. 8th)

15.6: #1, 3, 13, 15, 17, 19, 23

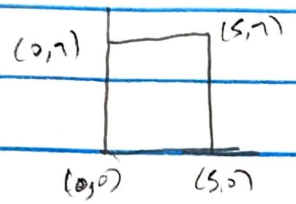
16.1: #1, 3, 5, 7, 9, 11, 17, 23, 25, 27

15.6: #1, 3, 13, 15, 17, 19, 23

1) a) on u-axis, $v=0$ on v-axis, $u=0$

image set = $\{(2u, u), (0, v)\}$

b) rectangle $R = [0,5] \times [0,7]$



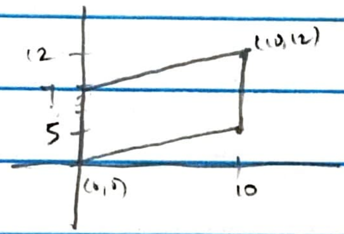
$G(0,0) = (0,0)$

$G(5,0) = (2\times5, 5+0)$

$= (10,5)$

$G(0,7) = (0,7)$

$G(5,7) = (10,12)$



$\boxed{(0,0),\ (10,5),\ (10,12)\ (0,7)}$

c) (1,2) and (5,3)

$$\frac{u-1}{2-1} = \frac{v-5}{3-5}$$

$$\frac{u-1}{1} = \frac{v-5}{-2}$$

$$v-5 = -2(u-1)$$

$$2u+v = 5+2$$

$$2u+v = 7$$

image of $2u+v=7$ under

$G(u,v) = (2u, u+v)$

$2(2u) + (u+v) = 7$

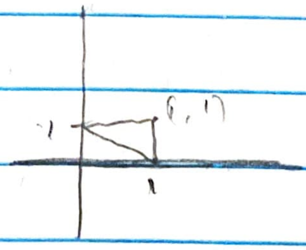
$5u+v = 7$

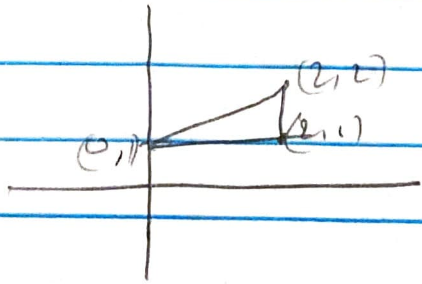
d) triangle with vertices (0,1), (1,0), (1,1)

$(1,0) \to (2,1)$

$(1,1) \to (2,2)$

$(0,1) \to (0,1)$





3) a) u-axis is $(x,y) = \phi(u,0) = (u^2, 0)$

$x = u^2,\ y = 0$

v-axis is $(x,y) = \phi(0,v) = (0,v)$

$x = 0,\ y = v$

b) $|u| \le 1,\ |v| \le 1$

$x = u^2,\ y = v$

$u = \pm\sqrt{x},\ v = y$, so $|\sqrt{x}| \le 1,\ |y| \le 1$

c) $0 \le u \le 1,\ v = u$

$u = \sqrt{x},\ v = y$

$0 \le \sqrt{x} \le 1,\ y = \sqrt{x}$

d)

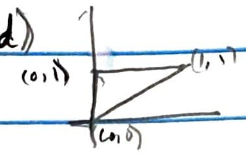


Image of OA is $u = 0,\ 0 \le v \le 1$

so $x = 0,\ 0 \le y \le 1$

13) $G(u,v) = (3u + 4v,\ u - 2v)$

$Jac(G) = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$Jac(G) = \begin{vmatrix} \frac{\partial}{\partial u}[3u+4v] & \frac{\partial}{\partial v}[3u+4v] \\ \frac{\partial}{\partial u}[u-2v] & \frac{\partial}{\partial v}[u-2v] \end{vmatrix}$

$Jac(G) = \begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix} = \boxed{-10}$

15) $G(r,t) = (r\sin t,\ r - \cos t)$; $(r,t) = (1, \pi)$

$Jac(G) = \begin{vmatrix} \frac{\partial}{\partial r} r\sin t & \frac{\partial}{\partial t} r\sin t \\ \frac{\partial}{\partial r} r - \cos t & \frac{\partial}{\partial t} r - \cos t \end{vmatrix}$

$Jac(G) = \begin{vmatrix} \sin t & r\cos t \\ 1 & \sin t \end{vmatrix}$

$Jac(G) = \sin^2 t - r\cos t$

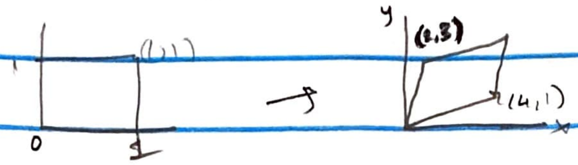
$Jac(1,\pi) = \sin^2 \pi - 1\cos \pi = \boxed{1}$

17) $G(r,\theta) = (r\cos\theta, r\sin\theta)$; $(r,\theta) = (4, \frac{\pi}{6})$

$$\text{Jac}(G) = \begin{vmatrix} \frac{\partial}{\partial r} r\cos\theta & \frac{\partial}{\partial \theta} r\cos\theta \\ \frac{\partial}{\partial r} r\sin\theta & \frac{\partial}{\partial \theta} r\sin\theta \end{vmatrix}$$

$$\text{Jac}(G) = r\cos^2\theta + r\sin^2\theta = \boxed{4}$$

19)



$\phi(u,v)$ be: $(Au+Bv, Cu+Dv)$

$\phi(0,1) = (B,D) = (2,3)$

$B=2, D=3$

$\phi(1,0) = (A,C) = (4,1)$

$\therefore A=4, C=1$

$$\boxed{\phi(u,v) = (4u+2v, u+3v)}$$

23) $J(\phi)\ \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} = -7$

$\text{Area}(\phi(R)) = |\text{Jac}\,\phi|\ \text{Area}(R)$

$= 7\ \text{Area}(R)$

a) Area of $R = [0,3] \times [0,5]$ is $3 \cdot 5 = 15$

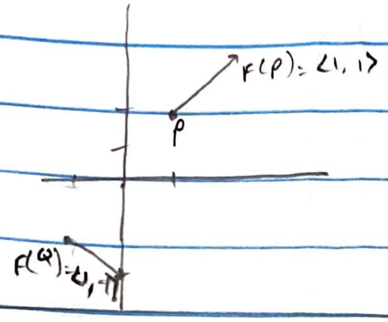
$7 \times 15 = \boxed{105}$

b) Area of $R = [2,5] \times [1,7]$ is $3 \cdot 6 = 18$

$\text{Area}(\phi(R)) = 7 \cdot 18 = \boxed{126}$

16.1: # 1, 3, 5, 7, 9, 11, 17, 23, ~~25~~, 27

1) $F(1,1) = \langle 1,1 \rangle$, $F(-1,-1) = \langle 1,-1 \rangle$

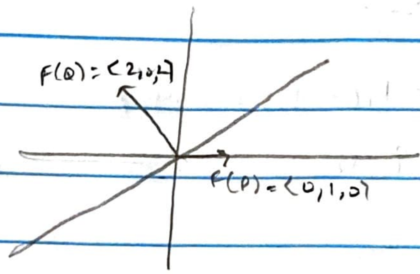


3) $P = (0,1,1)$ $Q = (2,1,0)$

$F = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$F(P) = \langle 0,1,0 \rangle$

$F(Q) = \langle 2,0,2 \rangle$



5) $F = \langle 1,0 \rangle$

$F = F_1\hat{i}$

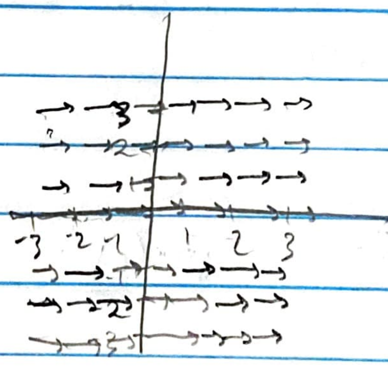
$F(x,y) = \langle 1,0 \rangle$

$= \hat{i}$

$-3 \le a \le 3$, $-3 \le b \le 3$

$F(a,b) = \langle 1,0 \rangle$

$r = \sqrt{a^2+b^2}$



7) $F = x\hat{i}$

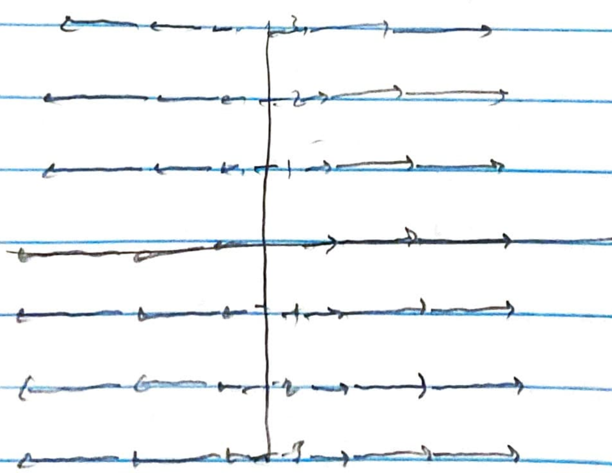
$F = F_1\hat{i} + F_2\hat{j}$

$F = x\hat{i} + 0\cdot\hat{i}$

$\langle x,0 \rangle$

$-3 \le a \le 3$, $-3 \le b \le 3$ $F(a,b) = \langle a,0 \rangle$

$r = \sqrt{a^2+b^2}$



9) $F = \langle 0, x \rangle$

$F = F_1\hat{i} + F_2\hat{j}$

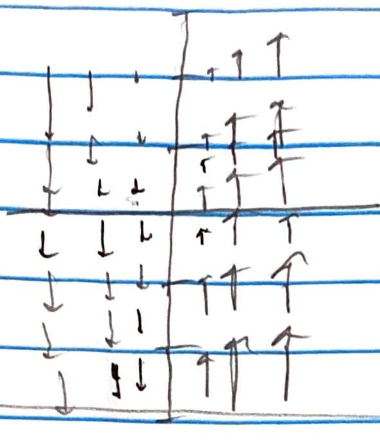
$F(x,y) = \langle 0, x \rangle$

$= x\hat{j}$

$-3 \le a \le 3,\ -3 \le b \le 3$

$F(a,b) = \langle 0, a \rangle$

$r = \sqrt{a^2+b^2}$



11) $F = \left\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right\rangle$

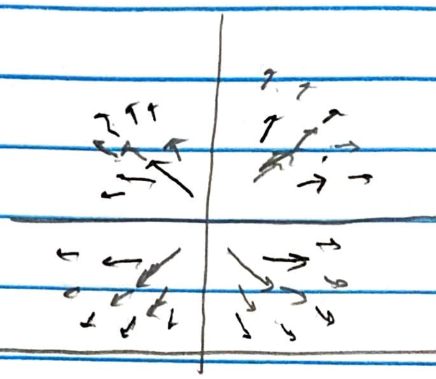
$F(x,y) = \langle F_1(x,y), F_2(x,y) \rangle$

$F = F_1\hat{i} + F_2\hat{j}$

$-3 \le a \le 3,\ -3 \le b \le 3$

$F(a,b) = \left\langle \frac{a}{a^2+b^2}, \frac{b}{a^2+b^2} \right\rangle$

$r = \sqrt{a^2+b^2}$



17) $F(x,y,z) = \langle 1, 1, 1 \rangle$

$F = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$F = \hat{i} + \hat{j} + \hat{k}$

$r = \sqrt{a^2+b^2+c^2}$ $\boxed{\text{plot (c)}}$

23) $\nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$$\nabla \times F \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 1+1+1 = \boxed{3}$

$\text{curl}(F) = \langle 0-0, 0-0, 0-0 \rangle = \boxed{\langle 0,0,0 \rangle}$

25) $F = \langle x - 2zx^2, z - xy, z^2x^2 \rangle$

$\nabla \cdot F \ \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$= \text{div}(F) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \underline{0}$

$\text{curl}(F) = \left\langle \frac{\partial F_3}{\partial y} + \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$

$= \underline{\langle 0, 0, 0 \rangle}$

27) $F(x, y, z) = \langle yz, xz, xy \rangle$

$\text{div}(F) = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$$\text{curl}(F) = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$

$\text{div}(F) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0 + 0 + 0 = \boxed{0}$

$\text{curl}(F) = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$

$= \boxed{\langle 0, 0, 0 \rangle}$